Algebraic Geometry

Notes by: Sara Lapan
Based on lectures given by: Karen Smith

Contents

[1]Books recommended for this course included, but were not limited to:
(1). *Basic Algebraic Geometry 1: Varieties in Projective Space* by Igor R. Shafarevich and M. Reid and
(2). *An Invitation to Algebraic Geometry* by Karen E. Smith, Lauri Kahanp, Pekka Keklinen, and William Traves

[2]These notes were typed during lecture and edited somewhat, so be aware that they are not error free. if you notice typos, feel free to email corrections to swlapan@umich.edu.

1. Introduction to Varieties:

1.1. Affine Varieties.

Lecture 1. Affine Algebraic Sets

Definition 1.1.1. Let k be a field. **Affine n-space**, $\mathbb{A}^n_k$, is a vector space of dimension n over k.

Our goal is to understand several types of algebraic varieties. Informally, an algebraic variety is a geometric object that looks locally like the zero set of a collection of polynomials.

Definition 1.1.2. An **affine algebraic set** is the common zero set, in $\mathbb{A}^n_k$, of a collection of polynomials $\{F_\lambda\}_{\lambda\in\Lambda}$, where $F_\lambda \in k[x_1, ..., x_n]$ and k is any field. This is denoted by $\mathbb{V}(\{F_\lambda\}_{\lambda\in\Lambda})$.

Examples:

(1) $\mathbb{V}(x^2 + y^2 + z^2 - 1) \subset \mathbb{C}^3$
(2) $\mathbb{V}(y - x^2) \subset \mathbb{A}^2_{\mathbb{C}}$
(3) $\{$All $n \times n$-matrices of determinant $1\} \subset \mathbb{A}^{n^2}$

Algebra Blackbox:

Definition 1.1.3. A ring is **Noetherian** if every ideal is finitely generated

Theorem 1.1.4. ***Hilbert Basis*** *If a ring R is Noetherian, then the polynomial ring over R in one variable, $R[x]$, is Noetherian.*

Remark 1.1.5. By finite induction on indeterminates, if R is a Noetherian ring, then $R[x_1, \ldots, x_n]$ is also Noetherian.

An immediate consequence of this algebraic result is:

Proposition 1.1.6. *Every affine algebraic set is the vanishing set of a finite collection of polynomials.*

Proof. Let $V \equiv \mathbb{V}(\{F_\lambda\}_{\lambda\in\Lambda}) \subset \mathbb{A}^n_k$. Let $I = (\{F_\lambda\}_{\lambda\in\Lambda}) \subset k[x_1, \ldots, x_n]$ so that $V = \mathbb{V}(I)$. k is Noetherian, so, by the Hilbert Basis theorem, $k[x_1, \ldots, x_n]$ is Noetherian. Therefore I is generated by finitely many polynomials. □

Lecture 2. Hilbert Nulltellensatz

The topology given to affine space is the Zariski-Topology. Instead of defining the topology in terms of open sets, it is defined in terms of closed sets. In particular, affine algebraic sets define the closed sets of the Zariski topology. The closed sets in the Zariski topology, excluding the space itself, are comparatively small since they are the zeros of finitely many polynomials.

Questions:

(1) *Is an arbitrary union of affine algebraic subsets of $\mathbb{A}^n$ an affine algebraic set?*
No. In $\mathbb{A}^1$ any closed set is given by an ideal. But every ideal is principal, so it is generated by one polynomial. Since a polynomial (non-zero) in one variable only has finitely many roots, all the closed sets in $\mathbb{A}^1$ are either empty, finite, or all of $\mathbb{A}^1$.
(2) *How does the Zariski-Topology (for closed sets) on $\mathbb{A}^2$ look?*
Curves in two variables, points, finite unions thereof, $\varnothing$, and $\mathbb{A}^2$ are all the closed sets.
(3) *Is the Zariski-topology Hausdorff?* No.

ALGEBRA BLACKBOX:

Definition 1.1.7. The **radical** of I is defined as: $RadI \equiv \{f \in R \mid f^n \in I \text{ for some } n\}$

Let $I \subseteq R$ be an ideal of the commutative ring with identity, R.

$$\{\text{(Radical) Ideals of } R/I\} \longleftrightarrow \{\text{ (radical) ideals in } R \text{ containing } I\}$$

One can think of affine algebraic sets as being given by an ideal of polynomials.
Let $\{F_\lambda\}_{\lambda\in\Lambda} \subseteq k[x_1, \ldots, x_n]$, then $\mathbb{V}(\{F_\lambda\}_{\lambda\in\Lambda}) = \mathbb{V}(I)$ where I is the ideal generated by $\{F_\lambda\}_{\lambda\in\Lambda}$.

Proposition 1.1.8. $\mathbb{V}(I) = \mathbb{V}(RadI)$

Proof. $I \subseteq RadI \Rightarrow \mathbb{V}(I) \supseteq \mathbb{V}(RadI)$. If $p \in \mathbb{V}(I)$ we want to show that $f \in RadI \Rightarrow f(p) = 0$. But for some $n \in \mathbb{N}$, $f^n \in I \Rightarrow f^n(p) = 0 \Rightarrow f(p) = 0 \Rightarrow p \in \mathbb{V}(RadI)$. □

Fix an affine algebraic set $\mathbb{V}(I) = V \subseteq \mathbb{A}^n$ (defined over some field k).

Definition 1.1.9. A **regular function** $g : V \to k$ is a function that agrees with the restriction of a polynomial.

Definition 1.1.10. The **coordinate ring of** V, $k[V]$, is the set of regular functions in V, considered with the obvious pointwise addition and multiplication of functions.

Example 1.1.11. Let $V = \mathbb{V}(xy-1) \subseteq k^2$. Then $\frac{1}{y} \in k[V]$ is a regular function, since $xy = 1$ on V implies that $\frac{1}{y} = x$ in V.

Remark 1.1.12. There is a natural ring homomorphism: $k[x_1, \ldots, x_n] \xrightarrow{\varphi} k[V]$ given by restriction, that is surjective and whose kernel is the ideal $\mathbb{I}(V) = \{G \in k[x_1, \ldots, x_n] \mid G(p) = 0 \ \forall\ p \in V\}$. Therefore, $k[V] \cong k[x_1, \ldots, x_n]/\mathbb{I}(V)$.

Properties of $k[V]$:

(1) It is a commutative, finitely generated k-algebra $\Rightarrow$ it is Noetherian
(2) It does not contain any non-zero nilpotent element (i.e. it is reduced). Equivalently, $\mathbb{I}(V)$ is radical.

Let $\mathcal{I}$ = collection of all radical ideals in $k[x_1, \ldots, x_n]$.
Let $\mathcal{V}$ = collection of all algebraic subsets of $\mathbb{A}^n$.

Remark 1.1.13. It is an easy fact that $\mathbb{V}(\mathbb{I}(V)) = V$, where V is a variety.

Proof. $V = \mathbb{V}(I)$ for some ideal $I \Rightarrow I \subseteq \mathbb{I}(V) \Rightarrow V \supseteq \mathbb{V}(\mathbb{I}(V))$. If $p \in V$ and $g \in \mathbb{I}(V)$, then $g(p) = 0$, so $p \in \mathbb{V}(\mathbb{I}(V)) \Rightarrow V \subseteq \mathbb{V}(\mathbb{I}(V))$. □

Theorem 1.1.14. ***(Hilbert's Nullstellensatz)*** *If k is algebraically closed and $I \subseteq k[x_1, \ldots, x_n]$ is an ideal, then $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$.*

Therefore there is a one-to-one order-reversing correspondence between affine algebraic sets in $\mathbb{A}^n$ and radical ideals in a polynomial ring when $k = \overline{k}$.

Geometric	$\xleftarrow{\mathbb{V}()}$ $\xrightarrow{\mathbb{I}()}$	Algebraic
$\mathbb{A}^n$		(0)
V		$\mathbb{I}(V) \supseteq (0)$
$W \subseteq V$		$\mathbb{I}(W) \supseteq \mathbb{I}(V)$
$p = (\lambda_1, \ldots, \lambda_n)$		$(x_1 - \lambda_1, \ldots, x_n - \lambda_n) \supseteq \mathbb{I}(W)$

Lecture 3. Developing Affine Varieties

Algebra Blackbox:

Definition 1.1.15. Let R be a ring. An ideal $P \subseteq R$ is **prime** if $xy \in P \Rightarrow x \in P$ or $y \in P$. Equivalently, P is prime $\Leftrightarrow R/P$ is a domain.

In a Noetherian ring, R, if an ideal $I \subseteq R$ is radical then it has a unique decomposition (up to order) into prime ideals containing it. In other words, $I = P_1 \cap \ldots \cap P_t$ for some prime ideals P_i.

Definition 1.1.16. A topological space V is **irreducible** if whenever $V = V_1 \cup V_2$, where $V_1, V_2 \subset V$ closed, then $V = V_1$ or $V = V_2$.

Proposition 1.1.17. $\mathbb{V}(I \cap J) = \mathbb{V}(I) \cup \mathbb{V}(J)$.

Proposition 1.1.18. *Fix an affine algebraic set $V \subseteq \mathbb{A}^n$.*
V is irreducible $\Leftrightarrow \mathbb{I}(V) \subseteq k[x_1, \ldots, x_n]$ is prime $\Leftrightarrow k[V]$ is a domain.

Proof. The second $\Leftrightarrow$ is clear since $k[V] \cong k[x_1, \ldots, x_n]/\mathbb{I}(V)$, so it remains to prove the first one. ($\Rightarrow$) Assume that V is reducible. Then $V = V_1 \cup V_2$ for some $V_i \subsetneq V$. Then $\mathbb{I}(V_i) \supsetneq \mathbb{I}(V)$ and $\mathbb{I}(V) = \mathbb{I}(V_1) \cap \mathbb{I}(V_2)$. Take $f_i \in \mathbb{I}(V_i) - \mathbb{I}(V)$, $f_1 f_2 \in \mathbb{I}(V) \Rightarrow \mathbb{I}(V)$ is not prime.
($\Leftarrow$) Take $f_1 f_2 \in \mathbb{I}(V) \equiv I$. We want to show that for f_1 or f_2 is in $\mathbb{I}(V)$. Assume not. Let $V_i \equiv \mathbb{V}(I, f_i) \subsetneq V$. Clearly $V \supseteq V_1 \cup V_2$. Take any $p \in V - V_1$, so $f_1(p) \neq 0$. Now $f_1 f_2 \in I$ so $f_1 f_2(p) = 0$ and I is prime $\Rightarrow f_2(p) = 0$. Therefore $p \in V_2$ and so $V = V_1 \cup V_2 \Rightarrow V$ is reducible. □

Theorem 1.1.19. *An affine algebraic set V has a unique decomposition (up to order) as a finite union of irreducible components.*

Proof. $V = \mathbb{V}(\mathbb{I}(V)) = \mathbb{V}(P_1 \cap \ldots \cap P_t) = \mathbb{V}(P_1) \cup \ldots \cup \mathbb{V}(P_t)$, where the P_i are prime ideals. □

Fix $V \subseteq \mathbb{A}^n, W \subseteq \mathbb{A}^m$ affine algebraic sets.

Definition 1.1.20. A **morphism** (or **regular map**) between affine algebraic sets $V \xrightarrow{f} W$ is a mapping (that is the restriction of) a polynomial map on the ambient affine spaces.

Example 1.1.21. If $V \xrightarrow{g} W$ such that $g = F|_V$, where $F = (F_1, \ldots, F_m)$ and $F_i \in k[x_1, \ldots, x_n]$, then g is a morphism.

Definition 1.1.22. An **isomorphism of affine algebraic sets** are morphism $V \underset{g}{\overset{f}{\rightleftarrows}} W$ such that their composition is the identity, $f \circ g = 1, g \circ f = 1$.

Crucial Concept:
Any regular map $V \xrightarrow{f} W$ induces a natural k-algebra homomorphism $k[W] \xrightarrow{f^*} k[V]$ called the pull-back, where $g \mapsto g \circ f$. This is functorial: If $V \xrightarrow{f} W \xrightarrow{g} W'$, then $k[V] \xleftarrow{f^*} k[W] \xleftarrow{g^*} k[W']$.

Lecture 4. Rational Functions

Algebra Blackbox:
Every domain R embeds into a (unique smallest) field $Q(R) = \{\frac{f}{g} \mid g \neq 0, f, g \in R\}$.

In fancy language, there is a contravariant functor between the category of affine algebraic sets and finitely generated reduced k-algebras. Assuming that $k = \overline{k}$, this functor defines an anti-equivalence of categories.

Theorem 1.1.23. *Fix an algebraically closed field, k:*

(1) *Every finitely generated, reduced algebra over k is (isomorphic to) the coordinate ring of some affine algebraic set, which is determined uniquely (up to isomorphism).*
(2) *Every k-algebra homomorphism between finitely generated reduced k-algebras is the pull-back of the corresponding regular map of the corresponding affine algebraic sets.*

Proof. (1) Fix R, a finitely generated reduced k-algebra. Fix a finite set of generators $y_1, \ldots, y_n$. This means that $k[x_1, \ldots, x_n] \twoheadrightarrow R$ sending $x_i \to y_i$ is a k-algebra homomorphism. Let I be the kernel of this map. I is radical since $k[x_1, \ldots, x_n]/I \cong R$. Let $V = \mathbb{V}(I) \subseteq \mathbb{A}^n, k[V] \cong k[x_1, \ldots, x_n]/\mathbb{I}(V)$. But $\mathbb{I}(\mathbb{V}(I)) = I$ since k is algebraically closed. The result follows.

(2) Let $R = k[y_1, \ldots, y_m]/J \xrightarrow{\varphi} S = k[x_1, \ldots, x_n]/I$. Defining this k-algebra map amounts to giving polynomials whose residue classes modulo I are the images of the y_i. So each $y_i \mapsto F_i \in k[x_1, \ldots, x_n]$. Consider $\mathbb{A}^n \xrightarrow{F=(F_1,\ldots,F_m)} \mathbb{A}^m$ when restricted to V, So $V \equiv \mathbb{V}(I) \xrightarrow{f \equiv F|_V} \mathbb{V}(J) \equiv W$. The coordinate ring of V (or (W) is S (or R). Since φ is well-defined, it must send J to I, so if $g \in J$, then $g \circ F \in I$.
Claim: f is a regular map whose pull-back recovers φ. First of all, $f(V) \subseteq W$: $\forall p \in V, F(p) \in W$. That is we need that for any $g \in J, g(F(p)) = g(F_1, \ldots, F_m)(p) = 0$, but $g \circ F \in I$, so $F(V) \subseteq W$.

□

Fix an irreducible affine algebraic set, V.

Definition 1.1.24. The **function field of V** (or the field of rational functions of V) is the quotient field $k(V)$ of $k[V]$.

Definition 1.1.25. A rational function φ is **regular at** $p \in V$ if $\exists f, g \in k[V]$ such that $g(p) \neq 0$ and $\varphi = \frac{f}{g}$.

Example 1.1.26. $V = \mathbb{V}(xz - yw) \subseteq \mathbb{A}^4, \varphi = \frac{x}{y}$ is regular at $(1,1,1,1)$ and $(1,1,0,0)$. Is it regular at $(0,0,1,1)$?
$\varphi = \frac{x}{y} = \frac{0}{0}$, but $xz - yw = 0 \Rightarrow \frac{x}{y} = \frac{w}{z} = 1$. So yes.

Definition 1.1.27. The locus of points where $\varphi \in k(V)$ is regular is the **domain of definition** of φ.

Proposition 1.1.28. *The domain of definition of $\varphi \in k(V)$ is a non-empty open subset of V in the Zariski-topology.*

Proof. Consider all possible representations, $\{\frac{f_\lambda}{g_\lambda}\}_{\lambda \in \Lambda}$, of φ. Clearly φ is regular on $U_\lambda = V - \mathbb{V}(g_\lambda)$, which is open, and so the domain of definition is $\cup_{\lambda \in \Lambda} U_\lambda$. □

Proposition 1.1.29. *The values of φ are uniquely determined everywhere on its domain of definition by its values on any (non-empty) open set.*

Remark 1.1.30. Let V be an irreducible variety. If $\varphi, \varphi' \in k(V)$ and $\varphi|_W = \varphi'|_W$ on some non-empty $W \subseteq V$ in the domain of definition of both functions, then their domains of definition are equal as are they.

Sketch of Proof. Let $\psi = \varphi - \varphi'$. It suffices to show that $\psi|_W = 0 \Rightarrow \psi = 0$ in $k(V)$. Take any representative $\psi = \frac{f}{g}, V = (V - W) \cup \mathbb{V}(f)$ (since $W \subseteq \mathbb{V}(f)$) gives a decomposition of V. It remains to show that V is not irreducible, which is a contradiction. □

Lecture 5. Introduction to Sheaf Theory

The following are definitions from Hartshorne's "Algebraic Geometry:"

Definition 1.1.31. Let X be a topological space. A **presheaf**, $\mathcal{F}$, of abelian groups on X consists of data:

- For every open subset $U \subseteq X$, an abelian group $\mathcal{F}(U)$, and
- For every inclusion $V \subseteq U$ of open subsets of X, a morphism of abelian groups $\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$

subject to the conditions:

(1) $\mathcal{F}(\emptyset) = 0$
(2) ρ_{UU} is the identity map $\mathcal{F}(U) \longrightarrow \mathcal{F}(U)$, and
(3) If $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

Definition 1.1.32. A presheaf $\mathcal{F}$ on a topological space X is a **sheaf** if it satisfies the following conditions:

(1) If U is an open set, $\{V_i\}$ is an open covering of U, and if $s \in \mathcal{F}(U)$ is an element such that $s|_{V_i} = 0 \forall i$, then $s = 0$. (Note that this condition implies that s is unique)
(2) If U is an open set, $\{V_i\}$ is an open covering of U, and if we have elements $s_i \in \mathcal{F}(V_i) \forall i$, with the property that for each i, j we have $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i \forall i$.

Definition 1.1.33. If $\mathcal{F}$ is a presheaf on X, and if P is a point of X, we define the **stalk** $\mathcal{F}_p$ of $\mathcal{F}$ at P to be the direct limit of the groups $\mathcal{F}(U)$ for all open sets U containing P via the restriction maps ρ.

Definition 1.1.34. If $\mathcal{F}$ and $\mathcal{G}$ are presheaves on X, a morphism of abelian groups $\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ for each open set U, such that whenever $V \subseteq U$ is an inclusion, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \Big\downarrow{\scriptstyle \rho_{UV}} & & \Big\downarrow{\scriptstyle \rho'_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

Fix $V \subseteq \mathbb{A}^n$ irreducible affine algebraic set of over $k = \overline{k}$.

Definition 1.1.35. Take an open set $U \subseteq V$. The **ring of regular functions on** U is the ring of all rationals functions that are regular on U. This is denoted by $\mathcal{O}_V(U) = \{\varphi \in k(V) \mid \varphi \text{ is regular at each } p \in U\}$.

$\mathcal{O}_V(U)$ is a domain because it is a subring of a domain, $k(V)$. This is a more local definition of the ring of regular functions than the previous definition.

Theorem 1.1.36. *If $U = V$, then $\mathcal{O}_V(U) = k[V]$*

Proof. Clearly $\mathcal{O}_V(U) \supseteq k[V]$ since if $f \in k[V]$, then $\frac{f}{1} \in \mathcal{O}_V(U)$. Take $\varphi \in k(V)$ that is regular $\forall p \in U = V$ (i.e. $\varphi \in \mathcal{O}_V(U)$). $\forall p \exists f_p, g_p \in k[V]$, such that $\varphi = \frac{f_p}{g_p}$ where $g_p(p) \neq 0$. Consider $\mathbb{V}(\{g_p\}_{p \in V}) = \varnothing \Rightarrow I(\mathbb{V}(\{g_p\}_{p \in V})) = (1)$. By Hilbert Nullstellensatz, $\exists h_1, \ldots h_t \in k[V]$ such that $1 = h_1 g_{p_1} + \ldots + h_t g_{p_t}$ (this comes from the result that $k[V] \cong k[x_1, \ldots, x_n]/\mathbb{I}(V)$). $\varphi = h_1(g_{p_1}\varphi) + \ldots + h_t(g_{p_t}\varphi) = h_1 f_{p_1} + \ldots h_t f_{p_t} \in k[V]$, since by definition $f_p = g_p \varphi$. Therefore $\mathcal{O}_V(U) \subseteq k[V]$. ◻

$\mathcal{O}_V$ maps open sets of V to k-algebras or rings. If $W \subseteq U$, then $\mathcal{O}_V(U) \xrightarrow[\text{restrict}]{\nu} \mathcal{O}_V(W)$.

Definition 1.1.37. A **presheaf of rings** (or groups, etc), $\mathcal{F}$, on a topological space X is a contravariant functor from the category of open sets of X to the category of rings (or groups, etc.).

We say that it is a sheaf if the elements of $\mathcal{F}(U)$ are like functions on U given by local properties. More precisely, if $U = \bigcup_{\lambda \in \Lambda} U_\lambda$ is open and $s_\lambda \in (F)(U_\lambda)$ with $s_\lambda|_{U_\lambda \cap U_{\lambda'}}$ then there is a unique $s \in \mathcal{F}(U)$ that restricts to s_λ on U_λ. This is the sheaf axiom.

Example 1.1.38. Smooth functions on differentiable manifolds, holomorphic functions (sheaf of rings), and sections on a k-vector bundle (sheaf of modules). Non-example: integrable functions on $\mathbb{R}^n$ form a presheaf but not a sheaf.

Remark 1.1.39. To every pre-sheaf there is a unique sheafification which makes it a sheaf. For instance, constant functions form a pre-sheaf and the sheafification makes it into a sheaf of locally constant functions.

Local Picture of Algebraic Geometry:
One can define an abstract algebraic variety over $k = \overline{k}$ to be a topological space, V, together with a sheaf of k-algebras (from $\mathcal{O}_V$) such that V has an open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ where U_λ is homeomorphic to an affine algebraic set and under this homeomorphism $\mathcal{O}_V|_{U_\lambda}$ becomes the sheaf of regular functions on V. Note that one could also define a differentiable manifold in this way.

1.2. **Projective Space.**

Lecture 6. Introduction to Projective Space

ALGEBRA BLACKBOX:

Proposition 1.2.1. *An ideal $I \subseteq k[x_1, \ldots, x_n]$ is homogeneous $\Leftrightarrow$ I can be generated by homogeneous elements $\Leftrightarrow$ If $f = \Sigma_{i=1}^n f_i \in I$ where f_i is a homogeneous polynomial of degree i, then each $f_i \in I$.*

Furthermore, if I is homogeneous, then $RadI$ is homogeneous and $k[x_1, \ldots, x_n]/I$ is graded.

Fix a field k (need not be algebraically closed) and a vector space V over k.

Definition 1.2.2. For any vector space V over k, the **projective space of** V, denoted $\mathbb{P}(V)$, is the set of all 1-dimensional subspaces of V. Alternatively one can write $\mathbb{P}_k^n = \mathbb{P}(k^{n+1})$ to denote the set of lines through the origin in k^{n+1}.

Example 1.2.3. $\mathbb{P}(\mathbb{C}^2)$: one usually thinks of this by fixing a reference line, for instance the line $z = 1$. Then $\mathbb{P}(\mathbb{C}^2)$ is every 1-dimensional subspace that intersects the reference line plus the one line that is parallel to the reference line. Equivalently, $\mathbb{P}(\mathbb{C}^2) = \mathbb{C} \cup \{\infty\}$.

Example 1.2.4. $\mathbb{P}_\mathbb{R}^2 = \mathbb{P}(\mathbb{R}^2) = \mathbb{R} \cup \mathbb{P}_\mathbb{R}^1$ and more generally, $\mathbb{P}_k^n = \mathbb{A}_k^n \cup \mathbb{P}_k^{n-1}$ where $\mathbb{P}_k^{n-1}$ is the stuff at infinity.

Goal: We want to think of $\mathbb{P}^n$ as something that looks locally like an algebraic variety $\mathbb{A}^n$, in fact, it is a natural compactification of $\mathbb{A}^n$.

We can represent a point $p \in \mathbb{P}_k^n = \mathbb{P}(k^{n+1})$ by a choice of basis so that $p = [x_0 : \ldots : x_n]$. We call the x_i homogeneous coordinates. Caution: the x_i are not well-defined functions on $\mathbb{P}^n$.
$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1}$ where $\mathbb{A}^n = \{p \mid p = [x_0 : \ldots : x_n], x_0 \neq 0\}$ and $\mathbb{P}^{n-1} = \{p \mid p = [0 : x_1 : \ldots : x_n]\}$. We get a standard affine cover of $\mathbb{P}^n$ from $\mathbb{A}^n$ given by open sets $U_i = \{p \mid p = [x_0 : \ldots : x_n], x_i \neq 0\}$.

Proposition 1.2.5. *Let $F \in k[x_0, \ldots, x_n]$ (so F is not a function on $\mathbb{P}^n$). If F is homogeneous of degree d, then $F(\lambda a_0, \ldots, \lambda a_n) = \lambda^d F(a_0, \ldots, a_n)$*

Definition 1.2.6. A **projective algebraic set** (projective variety) $V \subseteq \mathbb{P}^n_k$ is the common zero set of an arbitrary collection $\{F_\lambda\}_{\lambda\in\Lambda}$ of homogeneous polynomials in $k[x_0,\dots,x_n], V = \mathbb{V}(\{F_\lambda\}_{\lambda\in\Lambda}) \subseteq \mathbb{P}^n_k$.

Easy Facts:

(1) We can assume that $\{F_\lambda\}_{\lambda\in\Lambda}$ defining a projective algebraic set is finite.
(2) An arbitrary intersection of projective algebraic sets in $\mathbb{P}^n$ is also a projective algebraic set.
(3) If V, W are projective algebraic sets, then so is $V \cup W$.
(4) The sets of projective algebraic sets are the closed sets of a topology on $\mathbb{P}^n$ called the Zariski-topology.

Definition 1.2.7. Given a projective algebraic variety $V = \mathbb{V}(\{F_\lambda\}_{\lambda\in\Lambda}) \subseteq \mathbb{P}^n_k$, the **affine cone** over V is the affine algebraic set in $\mathbb{A}^{n+1}$ defined by the same polynomials.

Definition 1.2.8. The **homogeneous coordinate ring** $V \subseteq \mathbb{P}^n, k[V] = k[x_0,\dots,x_n]/\mathbb{I}(V)$, where $\mathbb{I}(V)$ is the ideal generated by homogeneous polynomials vanishing on V.

Remark 1.2.9. $\tilde{V} \subseteq \mathbb{A}^{n+1}$ affine cone of $V \subseteq \mathbb{P}^n$, then $k[V] \cong k[\tilde{V}]$ (this isomorphism is canonical).

Lecture 7. Introduction to Projective Space

$\mathbb{P}^n$ is covered by affine charts $\{U_i\}_i$, where $U_i = \{[x_0 : \cdots : x_n] \mid x_i \neq 0\}$ and is open since $U_i = \mathbb{P}^n - \mathbb{V}(x_i)$. These sets form the standard affine cover of $\mathbb{P}^n$ and a point $[x_0 : \cdots : x_n] \in U_i$ corresponds to
$(\frac{x_0}{x_i},\dots,\frac{x_n}{x_i}) \in \mathbb{A}^n$ where the i^{th} component is removed.

Fix $V = \mathbb{V}(F_1,\dots,F_t) \subseteq \mathbb{P}^n$ where the F_i are homogeneous polynomials in $x_0,\dots,x_n$.
V is covered by open sets $V \cap U_i \subseteq U_i$ and each of these sets is an affine algebraic variety in U_i. This concept is analogous to charts on manifolds.

Definition 1.2.10. If $V \subseteq \mathbb{A}^n$ is an affine algebraic set in $\mathbb{A}^n$, then its **projective closure** $\overline{V}$ is its Zariski-closure, in $\mathbb{P}^n$, under the embedding: $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$ by $(x_1,\dots,x_n) \mapsto [1 : x_1 : \cdots : x_n]$.

Example 1.2.11. Let $V = \mathbb{V}(xy-1) \subseteq \mathbb{A}^2 \subseteq \mathbb{P}^2, \mathbb{A}^2 \ni (x,y) \mapsto [x : y : 1] \in \mathbb{P}^2$.
Then $\overline{V} = V \cup \{[1:0:0] \cup [0:1:0]\}$, where these two points are identified at ∞. Can we homogenize this? Does $\mathbb{V}(xy-z^2) = \overline{V}$? Well,

$$\mathbb{V}(xy-z^2) \cap U_z = V \text{ and } \mathbb{V}(xy-z^2) \cap \text{"stuff at } \infty\text{"} = \mathbb{V}(xy-z^2) \cap \mathbb{V}(z) = \{[1:0:0] \cup [0:1:0]\}$$

So yes, we can homogenize $\mathbb{V}(xy-z^2)$.

Definition 1.2.12. If $f \in k[x_1,\dots,x_n]$, write $f = f_d + f_{d+1} + \cdots + f_{d+t}$ where the degree of f_i is i and $f_i \neq 0$. The **homogenization of degree** $d+t$ of f is $\tilde{f} \in k[x_0,\dots,x_n]$, where $\tilde{f} = x_0^t f_d + \cdots + x_0 f_{d+t-1} + f_{d+t}$.

Definition 1.2.13. If $F \in k[x_0,\dots,x_n]$ is homogeneous of degree t, we **de-homogenize** (with respect to x_0) F by setting $f(x_1,\dots,x_n) = F(1,x_1,\dots,x_n)$.

Theorem 1.2.14. *Let $V \subseteq \mathbb{A}^n$ be an affine algebraic set, $\overline{V}$ its projective closure. Then $\mathbb{I}(\overline{V}) \subseteq k[x_0,\dots,x_n]$ is generated by the homogenization of all elements of $\mathbb{I}(V)$.*

Proof. Let $\tilde{I} \subseteq k[x_0,\dots,x_n]$ be the ideal generated by the homogenization of all elements of $\mathbb{I}(V)$.
We want to show that $\tilde{I} = \mathbb{I}(V)$:
(⊆) $\tilde{f} \in \tilde{I}$ where $f \in \mathbb{I}(V) \Rightarrow \tilde{f}(1,x_1,\dots,x_n) = f \in \mathbb{I}(V) \Rightarrow \tilde{f}$ vanishes in $V \subseteq \tilde{V}$. So

$V \subseteq \mathbb{V}(f) \subseteq \mathbb{P}^n$ and $\overline{V} \subset \mathbb{V}(\tilde{f})$ because it is closed. $\Rightarrow \tilde{f} \in \mathbb{I}(\overline{V})$
($\supseteq$) Take any $F \in \mathbb{I}(\overline{V}) \Rightarrow F$ vanishes on $\overline{V} \Rightarrow V = \overline{V} \cap U_0 \Rightarrow F(1, x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$ vanishes on $V \Rightarrow f_i \mathbb{V} \Rightarrow F = x_0^q \tilde{f} \in \tilde{I}$ □

Example 1.2.15. Why you cannot homogenize randomly: Let $V = \{(t, t^2, t^3) \mid t \in k\} = \mathbb{V}(y - x^2, z - x^3) \subseteq \mathbb{A}^3$. Let $W = \mathbb{V}(wy - x^2, w^2 z - x^3) \subseteq \mathbb{P}^3$. Then $W \cap U_0 = \mathbb{V}(y - x^2, z - x^3) = V$, but W at ∞ is $W \cap \mathbb{V}(w) = \mathbb{V}(wy - x^2, w^2 z - x^3) = \mathbb{V}(w, x) = \{[0:0:y:z]\} \subseteq \mathbb{P}^1$. There's a point $[0:0:0:1]$ that is unaccounted for.

1.3. **Quasi-Projective Varieties:**

Lecture 8. Quasi-Projective Varieties

Let $L \subseteq \mathbb{A}^2$ be the line given by $y = mx + b$ for some fixed $m, b \in k$. We want to determine what point corresponds to ∞ on $\overline{L} \subset \mathbb{P}^2$. Equivalently, for any point $p \in L$ there is a line connecting p to the origin. The point $p \in L$ is sent to the point in $\mathbb{P}^2$ that corresponds to the line through the origin and p in $\mathbb{A}^2$.
We can solve this by using algebra: $\mathbb{P}^2$ has coordinates x, y, z. We need to homogenize the equation for L so that $\overline{L} = \mathbb{V}(y - mx - bz) \subseteq \mathbb{P}^2, \mathbb{V}(z) = \mathbb{P}^1$ at ∞. At $\infty, \mathbb{V}(y - mx - bz) \cap \mathbb{V}(z) = \mathbb{V}(y - mx, z) = [1 : m : 0]$, so we see that the line $y = mx$ in $\mathbb{A}^2$ goes to the point at infinity.
We can also solve this geometrically: the only line through the origin that does not intersect L is the line that is parallel to L, hence the line $y = mx$ must be the one corresponding to ∞ on L. In $\mathbb{P}^2$ this line corresponds to the point $[1 : m : 0]$.

Definition 1.3.1. A **quasi-projective variety** is a locally closed subset of $\mathbb{P}^n_k$.

Definition 1.3.2. A subset, V, of a topological space is **locally closed** in X if $V = U \cap C$ where $U \subset X$ is open and $C \subset X$ is closed.

Remark 1.3.3. Often when people say variety, they actually mean quasi-projective variety instead of an abstract variety.

For now, assume that $k = \overline{k}$ and that the subset X of $\mathbb{P}^n_k$ is a topological space with the induced Zariski-topology from $\mathbb{P}^n_k$.

Examples:

(1) Every projective algebraic set X is a quasi-projective variety
(2) Every affine algebraic set $X \subseteq \mathbb{A}^n \equiv U_0$ is a quasi-projective variety. $X = \overline{X} \cap U_0$
(3) Every open subset of a quasi-projective variety is a quasi projective variety.
(4) Every closed subset of a quasi-projective variety is a quasi-projective variety.

Now we need a notion of morphisms between quasi-projective varieties, but first we need a notion of regular functions of a quasi-projective variety. Fix a quasi-projective variety V/k. For each open set $U \subseteq V$, we want $\mathcal{O}_V(U)$ to be a k-algebra of regular functions on V.

(1) $\varphi : U \to k$ is an actual function such that φ is regular on $U \iff f$ is regular at each point in U
(2) $U_1 \subseteq U_2 \Rightarrow \mathcal{O}_V(U_2) \xrightarrow{restricts} \mathcal{O}_V(U_2)$ (note that this forms a sheaf)
(3) We want it to agree with the notion that we already have for irreducible affine algebraic sets.

Example 1.3.4. (*Quintessential*) Let $\varphi \equiv \frac{F(x_0, \ldots, x_n)}{G(x_0, \ldots, x_n)}$, where F and G are homogeneous polynomials of the same degree. This will be a regular function on $\mathbb{P}^n \setminus \mathbb{V}(G)$. Note that φ is well-defined because F, G are homogeneous of the same degree. $\mathbb{A}^n \cong U_i \equiv \mathbb{P}^n \setminus \mathbb{V}(x_i) \subseteq \mathbb{P}^n$ so that $\varphi|_{U_i}$ maps $(t_0, \ldots, \hat{t_i}, \ldots, t_n) \to [x_0 : \ldots 1 : \ldots : x_n]$ and is defined on $U_i \setminus \mathbb{V}(G)$.

Definition 1.3.5. Let $W \subseteq \mathbb{P}^n$ be a quasi-projective variety. A function $\varphi : W \to k$ is **regular** on W if for all points $p \in W$, there exists $F_p, G_p \in k[x_0, \ldots, x_n]$ homogeneous of the same degree such that φ agrees with the function $\frac{F_p}{G_p}$ on some neighborhood of p.

Remark 1.3.6. This gives us a sheaf of k-algebras $\mathcal{O}_V$ on every quasi-projective variety and it satisfies the three conditions. It also agrees with the definition of regular functions from before when we were restricting to affine varieties.

CAUTION: If V is an affine variety then $\mathcal{O}_V(V) = k[V]$ determines V completely, but this is not the case for a general quasi-projective variety. For instance, $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = k$

Quintessential examples of morphisms of quasi-projective varieties:

(1) $\mathbb{P}^1 \xrightarrow{\nu} \mathbb{P}^2$ by $[s : t] \to [s^2 : st : t^2]$ is a well-defined map since it is homogeneous. This function is given by regular functions in coordinates on affine charts.

$$\begin{array}{ccc} \mathbb{P}^1 \supseteq U_t \ni [s:1] & \to & \mathbb{A}^1 \ni s \\ \Downarrow & & \Downarrow \\ \mathbb{P}^2 \supseteq U_z \ni [x:y:z] & \to & \mathbb{A}^2 \ni (x,y) \end{array}$$

The image of $\nu = \mathbb{V}(y^2 - xz)$

(2) Let $\mathbb{P}^2 \supset V \equiv \mathbb{V}(z - y^2) \xrightarrow{\pi} \mathbb{P}^1$, where the map π sends $[x : y : z]$ to $\begin{cases} [x:y], & \text{if } x \neq 0 \\ [y:z], & \text{if } z \neq 0 \end{cases}$

We need to check that this map is well-defined (i.e. scaling, anything sent to 0, agrees on overlaps). $V \cap U_z = \mathbb{V}(x - y^2), V \cap U_y = \mathbb{V}(xz - 1)$. Note that ν, π are inverses.

Lecture 9. The Veronese Map

Definition 1.3.7. A **morphism (regular map)** $\varphi : X \subset \mathbb{P}^N \to Y \subset \mathbb{P}^M$ of quasi-projective varieties is a map of sets which is locally given by regular functions on affine charts. More precisely, $\forall x \in X, \exists$open $U \subseteq X$ containing x such that $\varphi(U) \subseteq U_i$ and on U, $\varphi : p \in U \to (\varphi_1(p), \ldots, \varphi_M(p)) \in Y \cap U_i \subseteq U_i = \mathbb{A}_M$, where $\varphi_i \in \mathcal{O}_X(U)$ are regular functions on U.

Definition 1.3.8. An **isomorphism of quasi-projective varieties** is a morphism $\varphi : X \to Y$ that admits a (regular morphism) inverse $\psi : Y \to X$ such that $\varphi \circ \psi : Y \to Y = 1_Y$ and $\psi \circ \phi = 1_X$.

CAUTION: Two projective varieties can be isomorphic and still have non-isomorphic coordinate rings.

Example 1.3.9. Consider the projection from a point $p \in \mathbb{P}^2$ onto a line l and send $q \in \mathbb{P}^2 \setminus \{p\}$ to $\overline{qp} \cap l \in l$. To show that this map is regular we choose a convenient basis so let $p = [0 : 1 : 0]$ and $l = \mathbb{V}(y)$. Let $q = [a : b : c]$, then $\overline{pq}$ is the 2-dimensional subspace spanned by p and q whereas l is the 2 dimensional subspace spanned by $[1 : 0 : 0]$ and $[0 : 0 : 1]$. $l \cap \overline{pq} = [a : 0 : 0]$ so $[x : y : z] \mapsto [x : 0 : z]$. We could have chosen coordinates so that l was the line at infinity, corresponding to $\mathbb{V}(z)$, and $p = [0 : 0 : 1]$. then $\pi(q)$ will be the point at ∞ corresponding to the slope of $\overline{pq}$.

Veronese Maps:
Fix N, d, M (where $M = \binom{N+d}{d} - 1$). Define the Veronese map ν_d as follows:

$$\mathbb{P}^N \xrightarrow{\nu_d} \mathbb{P}^M \text{ where } [x_0 : \cdots : x_N] \mapsto [x_0^d : x_0^{d-1} x_1 : \cdots : x_N^d]$$

Note that all of the terms in the image are monomials in $x_0, \dots, x_N$ of degree d.
Also, we can think of $\mathbb{P}^M$ as $\mathbb{P}((k[x_0, \dots, x_N])^d) = \mathbb{P}(Sym^d V)$, where V has dimension N.
We have already seen the Veronese map in early examples.

Example 1.3.10. $\mathbb{P}^1 \xrightarrow{\nu_3} \mathbb{P}^3$ where $[s:t] \mapsto [s^3 : s^2 t : st^2 : t^3]$. The image of ν_d is twisted cubic.

In general, the image of ν_d is a "twisted degree d curve" or a "rational (normal) curve of degree d."

Example 1.3.11. $\mathbb{P}^2 \xrightarrow{\nu_2} \mathbb{P}^5$, where $[s:t:u] \mapsto [s^2 : t^2 : u^2 : su : st : ut]$. It is nice to use the coordinates z_{ijk} for the monomial $s^i t^j u^k$ in this scenario (this, of course, easily generalizes).

Proposition 1.3.12. $\mathbb{P}^N \xrightarrow{\nu_d} \mathbb{P}^M$ *defines an isomorphism between* $\mathbb{P}^N$ *and a closed set in* $\mathbb{P}^M$.

Sketch of Proof. Let $W \subset \mathbb{P}^M$ be the closed set $\{z_I \mid I = (i_0, \dots, i_N) \text{ and } |I| = d\}$.
Then $W = \mathbb{V}(\{z_I z_J - z_k z_L\}_{I+J+K+L=d})$. Check that $W = Im \nu_d$. Also the inverse is just the projection $[x_0^d : x_0^{d-1} x_1 : \cdots : x_N^d] \in \mathbb{P}^m \mapsto [x_0 : x_1 : \cdots : x_N]$, where $x_0^d \neq 0$. ☐

Lecture 10. The Segre Map

Algebra Blackbox:
If $A \in M_{m \times n}(k)$, then the following are equivalent:
(1) The row space of A has dimension at most t
(2) The column space of A has dimension at most t
(3) The $(t+1)$ minors vanish
(4) $A = BC$ where $B \in M_{m \times t}(k)$ and $C \in M_{t \times n}(k)$

Any linear invertible linear transformation $V \xrightarrow{T} V$, where $V \cong k^{n+1}$, always induces a morphism $\mathbb{P}(V) \longrightarrow \mathbb{P}(V)$ by sending $[x_0 : \cdots : x_n] \mapsto [L_0(\underline{x}) : \cdots : L_n(\underline{x})]$ where $L_i \in V^*$.
In fact, the set of all invertible regular maps $\mathbb{P}^n \longrightarrow \mathbb{P}^n$ is $Aut(\mathbb{P}^n) = PSL(n+1)$.

Definition 1.3.13. A polynomial, $F(x_0, \dots, x_n, y_0, \dots, y_m)$ is **bihomogeneous** if it is homogeneous in the x_i's and homogeneous in the y_i's.

The Segre Map:
This is a map from the product of two projective spaces to a projective space.
For example, define $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sigma_{11}} \mathbb{P}^3$, where the coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$ are s, t, u, v and on $\mathbb{P}^3$ are w, x, y, z, as: $([s:t],[u:v]) \mapsto [su : sv : tu : tv] = [w : x : y : z]$. This map is well-defined.

Proposition 1.3.14. *The above map,* σ_{11}*, defines a bijection between* $\mathbb{P}^1 \times \mathbb{P}^1$ *and* $\Sigma \equiv \mathbb{V}(wz - xy)$.
Note that $\Sigma = \mathbb{V}(det \begin{pmatrix} w & x \\ y & z \end{pmatrix}) = \{rank\ 1,\ 2\ by\ 2\ matrices\}$.

Sketch of Proof. We can define the map $\Sigma \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ as:

$$[x:y:z:w] \mapsto \begin{cases} ([w:x],[w:y]) & w \neq 0 \\ ([w:x],[x:z]) & x \neq 0 \\ ([y:z],[x:z]) & z \neq 0 \\ ([y:z],[w:y]) & y \neq 0 \end{cases}$$

Check that these formulas do in fact agree on their overlaps. Also note that the map $\Sigma \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is equivalent to:

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} \mapsto \text{(projection onto row, projection onto column)}$$

Look in affine chart:

$$\mathbb{P}^1 \times \mathbb{P}^1 \supseteq \mathbb{A}^1 \times \mathbb{A}^1 \ni ([1:t],[1:v]) \mapsto [1:v:t:tv] \in \mathbb{A}^3 = \mathbb{V}(wz - xy) \subseteq \mathbb{P}^3$$

$$\text{Fix } b = [1:t] \text{ so that } b \times \mathbb{A}^1 \longrightarrow \mathbb{A}^3 \text{ has the image } \{(v, b, bv)\} = \{ \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} + v \begin{pmatrix} 1 \\ 0 \\ b \end{pmatrix} \}_{v \in k}$$

Note that the image is a set of lines in $\mathbb{A}^3$. So Σ is covered by families of disjoint lines in $\mathbb{P}^3$, $\{\sigma_{11}(b \times \mathbb{P}^1)\}_{b \in \mathbb{P}^1}$ and, by symmetry, another disjoint family of lines $\{\sigma_{11}(\mathbb{P}^1 \times b)\}_{b \in \mathbb{P}^1}$. Think of Σ as $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in a natural way in $\mathbb{P}^3$. Look at Σ at ∞ in $\mathbb{P}^3$. $\Sigma \cap U_w$ is the finite part. At $\infty, \Sigma \cap \mathbb{V}(w) = \mathbb{V}(wz - xy, w) = \mathbb{V}(xy, w) = \mathbb{V}(x, w) \cup \mathbb{V}(y, w) \subseteq \mathbb{P}^3$. □

Remark 1.3.15. It is easy to show that Σ is the product of $\mathbb{P}^1 \times \mathbb{P}^1$ on the category of projective varieties.

General construction of the Segre Map: Define $\sigma_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^{(m+1)(n+1)-1}$ by:

$$([x_0 : \cdots : x_n], [y_0 : \cdots : y_m]) \mapsto \{ \text{ the entries of the } (m+1) \times (n+1) \text{ matrix}\}$$

This matrix is generated by: $\begin{pmatrix} x_0 \\ \vdots \\ x_m \end{pmatrix} \begin{pmatrix} y_0 & \dots & y_n \end{pmatrix}$. Call the coordinates of $\mathbb{P}^{(m+1)(n+1)-1}$, $\{z_{ij}\}_{i,j}$ where $i = 0, \dots, n$ and $j = 0, \dots, m$. Define the image of $\sigma_{m,n}$ to be $\Sigma_{m,n} \subseteq \mathbb{P}^{(m+1)(n+1)-1} = \mathbb{P}((m+1) \times (n+1) \text{ matrices})$.

There are projections of $\Sigma_{m,n}$ to $\mathbb{P}^m$ and $\mathbb{P}^n$ by projection onto the row or column of A, respectively, where $A \in \Sigma_{m,n}$. $\Sigma_{m,n}$ is covered by a family of disjoint linear spaces $\{\sigma_{m,n}(\mathbb{P}^m \times \{p\})\}_{p \in \mathbb{P}^n}$ and $\{\sigma_{m,n}(\{p\} \times \mathbb{P}^n)\}_{p \in \mathbb{P}^m}$.

Lecture 11. Projective Hypersurfaces

ALGEBRA BLACKBOX:

Proposition 1.3.16 (Eisenstein's Criterion)**.** *Let P be a prime ideal of the integral domain R and let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial in $R[x]$ (here $n \geq 1$). Suppose $a_{n-1}, \dots, a_1, a_0$ are the elements of P and suppose a_0 is not an element of P^2. Then $f(x)$ is irreducible in $R[x]$.*

Remark 1.3.17. Eisenstein's Criterion is a very useful tool for determining if polynomials are irreducible.

Remark 1.3.18. There is a corrrespondence between $Sym^d(V^*)$ and homogeneous degree d polynomials on k^{n+1} by using $x_0^{a_0} \dots x_n^{a_n}$ as a basis where $\Sigma a_i = d$.

Theorem 1.3.19. ***(Thm 1)*** *The image of a projective variety under a regular mapping is closed (in fact it is projective). More precisely, if $X \xrightarrow{\varphi} Y$ is a morphism of quasi-projective varieties with X projective, then the image of X in Y is closed in Y.*

Remark 1.3.20. There is nothing like this for affine varieties.

The following is an analogous theorem in topology:

Theorem 1.3.21. *If* $X \xrightarrow{\varphi} Y$ *is a continuous map of topological spaces with* X *compact, then* $\varphi(X)$ *is compact.*

Corollary 1.3.22. *If* X *is a projective and connected variety (e.g. irreducible) then the only regular functions on* X *are the constant functions* $\mathcal{O}_X(X) = k$*.*

Proof. Take a regular map $\varphi \in \mathcal{O}_X(X)$, so $X \xrightarrow{\varphi} k = \mathbb{A}^1 \subseteq \mathbb{P}^1$. X is projective implies that $Im\varphi$ is closed in $\mathbb{A}^1$, which is closed in $\mathbb{P}^1$. All closed sets in $\mathbb{P}^1$ are either empty, finite, or all of $\mathbb{P}^1$. Therefore $Im\varphi$ is a finite set. But X is connected $\Rightarrow Im\varphi =$ one point. □

Corollary 1.3.23. *If* $X \xrightarrow{\varphi} Y$ *is a regular morphism from a connected projective variety to an affine variety, then* $\varphi(X)$ *is a point in* Y*.*

Proof. $X \longrightarrow Y \subseteq \mathbb{A}^m$ by sending $x \mapsto (\varphi_1(x), \ldots, \varphi_m(x))$. Each φ_i is a regular function on $X \Rightarrow \varphi_i(x) = \lambda_i \forall x($ by the previous corollary$) \Rightarrow \varphi(x) = (\lambda_1, \ldots, \lambda_m)$. □

Corollary 1.3.24. *There are no projective varieties sitting inside of affine varieties except for points.*

General Phenomenon: The set of varieties you want to study often forms a variety in some natural way. **Thm 1** will underlie the intuition that the BAD OBJECTS in that set form a small subset. The bad set is a proper Zariski closed set of the variety of the varieties that you are studying.

Example 1.3.25. Let $\mathbb{P}^n$ have coordinates $x_0, \ldots, x_n$. The set of hyperplanes in $\mathbb{P}^n$ forms a variety in a natural way.

$$H = \mathbb{V}(a_0x - 0 + \cdots + a_nx_n) \subseteq \mathbb{P}(V) \longleftrightarrow [a_0 : \cdots : a_n] \in \mathbb{P}^n = \mathbb{P}(V^*)$$

Fix a point $p = [\lambda_0 : \cdots : \lambda_n]$. When is $p \in H$?

$$\begin{aligned} p \in H &\Leftrightarrow a_0\lambda_0 + \cdots + a_n\lambda_n = 0 \\ &\Leftrightarrow H \in \mathbb{V}(a_0\lambda_0 + \cdots + a_n\lambda_n) \subseteq \mathbb{P}(V^*) \end{aligned}$$

We say that the "general hyperplane in $\mathbb{P}^n$ does not pass through p."

Definition 1.3.26. A **hypersurface of degree** d in $\mathbb{P}^n$ is the zeroset of a single degree d, homogeneous polynomial in $(n+1)$-variables.

Definition 1.3.27. A **conic** is a degree two hypersurface in $\mathbb{P}^2$.

Let $C = \mathbb{V}(ax^2 + bxy + cxz + dy^2 + eyz + fz^2) \subseteq \mathbb{P}^2$. Then C is a conic and there is a correspondence between C and $\mathbb{P}^5$ given by:
$C = \mathbb{V}(ax^2 + bxy + cxz + dy^2 + eyz + fz^2) \leftrightarrow [a : b : c : d : e : f] \in \mathbb{P}^5 = \mathbb{P}(Sym^2(V^*))$.
Ths set includes degenerate conics, for example $\mathbb{V}(xy)$ which looks like +, $\mathbb{V}(x^2)$ which is a double line, and $\mathbb{V}(x(x + \lambda y))$ which is a pair of intersecting lines.

Remark 1.3.28. In general, the set

$$\{\text{set of all degree d hypersurfaces in } \mathbb{P}(V), \text{ including degenerate ones}\} = \mathbb{P}(Sym^d(V^*))$$

.

Theorem 1.3.29. *The subset of reducible degree d hypersurfaces in* $\mathbb{P}(V)$ *is a (proper) closed subset of the parameter space* $\mathbb{P}(Sym^d(V^*))$ *of all hypersurfaces of degree d.*

Proof. Let F be a homogeneous polynomial of degree d.
$\mathbb{V}(F)$ is irreducible $\Leftrightarrow$ F does not factor as $F = F_tF_{d-t}$, where F_i is a homogeneous degree i polynomial.
Let $R = \cup_{i=1}^{d-1} R_i =$ the subset of all reducible degree d hypersurfaces $\subseteq \mathbb{P}(Sym^d(V^*))$,

where $R_i = \{\mathbb{V}(F) \in R \mid F \text{ factors as } F_iF_{d-i}\}$. Show that each R_i is closed in $\mathbb{P}(Sym^dV^*)$.
Consider $\mathbb{P}(Sym^iV^*) \times \mathbb{P}(Sym^{d-i}V^*) \longrightarrow \mathbb{P}(Sym^dV^*)$ where $(F,G) \mapsto FG$. The image of this map is closed in $\mathbb{P}(Sym^dV^*)$, R_i is the image $\Rightarrow R_i$ is closed. □

Lecture 12.

We are building towards proving that projective varieties are proper (proper is the analog to compact)

Theorem 1.3.30. *If X is a projective variety and Y is a quasi-projective variety, then $X \times Y \xrightarrow{project} Y$ is a closed map.*

Remark 1.3.31. Use the Segre embedding to determine the topology on $X \times Y$. You could also find the topology on a product by doing it patch by patch with open sets identified with $\mathbb{A}^n$ and then gluing together all of the sets. Don't use the product topology!

We can associate the hypersurfaces of degree d in $\mathbb{P}(V)$ to points in $\mathbb{P}(Sym^d(V^*))$. The irreducible (i.e. non-degenerate) hypersurfaces of degree d in $\mathbb{P}(V)$ are an open set of hypersurfaces of degree d in $\mathbb{P}(V)$.

Example 1.3.32. **Associating conics in $\mathbb{P}^2$ with $\mathbb{P}^5$:**

$$Q(x,y,z) \equiv ax^2 + by^2 + cz^2 + dxy + exz + fyz = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & \frac{d}{2} & \frac{c}{2} \\ \frac{d}{2} & b & \frac{f}{2} \\ \frac{e}{2} & \frac{f}{2} & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Let $C = \mathbb{V}(Q)$. By sending the polynomial Q to $[a : b : c : d : e : f]$, we get a map from a conics in $\mathbb{P}^2$ tos $\mathbb{P}^5$.

Digression on Products and Topologies:
Recall we have the Segre embedding: $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$.
If the coordinates are $(x_i, y_j) \in \mathbb{P}^n \times \mathbb{P}^m$ and $z_{ij} \in \mathbb{P}^{(n+1)(m+1)-1}$, then $(x_i, y_j) \to x_iy_j = z_{ij}$.
$V = \mathbb{V}(G_\alpha(z_{ij})) \cap \mathbb{P}^n \times \mathbb{P}^m$ where G_α is a homogeneous polynomial of degree d_α.
So $G_\alpha(x_iy_j)$ is a bihomogeneous polynomial of degree (d,d).

Example 1.3.33. $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ with coordinates $x_0, x_1, y_0, y_1, z_{00}, z_{01}, z_{10}, z_{11}$

Now, say $F(x_i, y_j)$ is bihomogeneous of degree $(r,s), r > s$. Then: $\mathbb{V}(F(x_iy_j)) = \mathbb{V}(y_0^{r-s}F(x_iy_j), \ldots, y_m^{r-s}F(x_iy_j))$.
Conclusion:

1. Every closed set in $\mathbb{P}^n \times \mathbb{P}^m$ has the form:
   $\mathbb{V}$(a bunch of bi-homogeneous polynomials, not necessarily bi-degree)
2. Every closed subset of $\mathbb{P}^n \times \mathbb{A}^m$, where $\{x_i\}$ are the variables in $\mathbb{P}^n$ and $\{y_j\}$ in $\mathbb{A}^n$, has the form:
   $\mathbb{V}$(a bunch of polynomials which are homogeneous in the x_i's).

Let $f : \mathbb{P}^n \to \mathbb{P}^m$ be given by $(x_i) \to (f_j(x_i))$ where the f_j's are homogeneous polynomials of degree d. $\Gamma_f \equiv \{(x,y) \mid f(x) = y\} \subseteq \mathbb{P}^n \times \mathbb{P}^m$. How do we check that Γ_f is closed?

$$\Gamma_f = \mathbb{V}\left(2 \times 2 \text{ minors of: } \begin{pmatrix} f_0(x_i) & f_1(x_i) & \ldots & f_m(x_i) \\ y_0 & y_1 & \ldots & y_m \end{pmatrix} \right)$$

This is nondegenerate $\Rightarrow$ the determinant of the matrix is nonzero.

If $X = \mathbb{V}(\{f_\alpha\}) \subseteq \mathbb{P}^n, Y = \mathbb{V}(\{g_\beta\}) \subseteq \mathbb{P}^m$, is $X \times Y$ closed in $\mathbb{P}^n \times \mathbb{P}^m$? Yes, $X \times Y = \mathbb{V}(f_\alpha, g_\beta)$.

Question: How many points determine a conic in $\mathbb{P}^2$?

Note that two points in projective space determine a line and that conics can be identified with $\mathbb{P}^5$.
Fix $p \in \mathbb{P}^2$. How many conics go through p?
Let $p = [p_0 : p_1 : p_2]$ and $Q(x, y, z) = ax^2 + by^2 + cz^2 + \ldots$.
Then $Q(p_0, p_1, p_2) = ap_0^2 + bp_1^2 + cp_2^2 + \ldots$.
So we have $\{$conics through $p\} \subseteq \{$conics$)$ $\} \to H$ hyperplane $\simeq \mathbb{P}^4 \subseteq \mathbb{P}^5$ closed.
Say that we have 5 points in $\mathbb{P}^2$ with no 4 points on a line. Then the intersection of the 5 hyperplanes through those points is a single point in $\mathbb{P}^5$.

Claim: There is only one conic through these points.

Proof. Suppose there are 2 such conics C_1, C_2 where $C_i = \mathbb{V}(Q_i)$ and C_1 is nondegenerate. We may assume that $Q_1(x, y, z) = xz - y^2$ since all nondegenerate conics are isomorphic (by problem set). $\mathbb{V}(Q_1) = Im(\nu_2)$. Look at $Q_2(s^2, st, t^2)$, which is a homogeneous polynomial of degree 4. If $Q_2(s^2, st, t^2)$ is not identically zero, then it has at most 4 zeros. But it has 5 zeros!
Exercise: look at the case when Q_1 is degenerate. □

Lecture 13. Closed Maps

Proposition 1.3.34. *If $Y \subseteq \mathbb{P}^m$ is a quasi-projective variety then $Y = \cup Y_i$ where $Y_i \subseteq Y$ open and Y_i is affine. Furthermore, $Z \subseteq Y$ is closed $\iff$ $Z \cap Y_i$ is closed in each Y_i.*

Proof. $Y = X \cap U$ for some $X \subseteq \mathbb{P}^m$ closed and $U \subseteq \mathbb{P}^m$ open $\Rightarrow Y = \bigcup_{i=0}^{n}(X \cap U_i \cap U)$, where $U_i = \mathbb{P}^n - \mathbb{V}(x_i)$. Let $X_i = X \cap U_i \Rightarrow X_i \cap U$ is open in X_i. Since $X_i \cap U$ is open in X_i and X_i is open in $X \Rightarrow X_i \cap U$ is open in Y. If we let $Y_i = X \cap U_i \cap U$, then our claim is proven. □

Theorem 1.3.35. ***A:*** *If X is projective, Y is quasi-projective, and $X \xrightarrow{\varphi} Y$ is a morphism of quasi-projective varieties, then $\varphi(X)$ is closed in Y. In other words, φ is a closed map.*

Theorem 1.3.36. ***B:*** *If X is projective and Y is quasi-projective then the projection $X \times Y \xrightarrow{\pi} Y$ is a closed map.*

Proof. **Thm B $\Rightarrow$Thm A:** Given φ as in Thm A, let $\Gamma_\varphi = \{(x, \varphi(x)) \mid x \in X\}$ be its graph. By a previous theorem, Γ_φ is closed in $X \times Y$. Therefore, by Thm B, $\pi(\Gamma_\varphi) = \varphi(X) \subseteq Y$ is closed.
Proof of Thm B:
Step 1: We want to reduce this to the case when $X = \mathbb{P}^n$ and $Y = \mathbb{A}^m$.
Claim: If the theorem is true for $\mathbb{P}^n$ then it is true for any projective variety X.
Assuming that $\mathbb{P}^n \times Y \xrightarrow{\pi} Y$ is a closed map, suppose that $X \subseteq \mathbb{P}^n$ and $Z \subseteq X \times Y$ is closed. Since Z is closed in $X \times Y$ and $X \times Y$ is closed in $\mathbb{P}^n \times Y$, Z is closed in $\mathbb{P}^n \times Y$. Therefore $\pi(Z)$ is closed. This argument is illustrated by the following diagram.

$$\begin{array}{ccc}
\mathbb{P}^n \times Y & \xrightarrow{\pi} & Y \\
\text{closed} \uparrow \subseteq & & = \uparrow \\
X \times Y & \longrightarrow & Y \\
\text{closed} \uparrow \subseteq & & \subseteq \uparrow \\
Z & \xrightarrow{\pi} & \pi(Z)
\end{array}$$

Therefore if theorem B is true for $\mathbb{P}^n$, it must be true for any projective variety $X \subseteq \mathbb{P}^n$.
Claim: If the theorem is true for $\mathbb{A}^m$ then it is true for any affine variety Y.
We can cover Y by affine sets, Y_i, so that $Y = \cup_i Y_i$. Let $Z \subseteq \mathbb{P}^n \times Y$. Then $\mathbb{P}^n \times Y_i \longrightarrow Y_i$ and $\mathbb{P}^n \times Y \supseteq Z \cap (\mathbb{P}^n \times Y_i) \longrightarrow \pi(Z) \cap Y_i$ where both sets are closed.
Now look at: $\mathbb{P}^n \times Y \xrightarrow{\pi} Y$ by $Z \mapsto \pi(Z)$. $\pi(Z)$ is closed in $Y \Leftrightarrow Z \cap (\mathbb{P}^n \times Y_i)$ is closed in $Y_i \forall i$. The latter we just saw is true, so $\pi(Z)$ is closed in Y. Therefore if theorem B is true for $\mathbb{A}^m$, it must be true for any quasi-projective variety Y.
Hence we may assume that $X = \mathbb{P}^n$ and $Y = \mathbb{A}^m$.
Step 2: We want to show that $\mathbb{P}^n \times \mathbb{A}^m \xrightarrow{\pi} \mathbb{A}^n$ is a closed map.
Any closed set in $\mathbb{P}^n \times \mathbb{A}^m$ is of the form $Z \equiv \mathbb{V}(F_1(x_0, \ldots, x_n, y_0, \ldots, y_m), \ldots)$ (note that F_i is homogeneous in x). We want to show that $\pi(Z) \subseteq \mathbb{A}^n$ is closed.

$$\begin{aligned}
\pi(Z) &= \{(\lambda_0, \ldots, \lambda_m) \in \mathbb{A}^m \mid \pi^{-1}(\lambda_0, \ldots, \lambda_m) \neq \varnothing\} \\
&= \{\underline{\lambda} \in \mathbb{A}^m | \mathbb{V}(F_1(x, \lambda), \ldots) \neq \varnothing\} \\
&= \{(\underline{\lambda} \in \mathbb{A}^m \mid Rad(F_1(\underline{X}, \lambda), \ldots) \not\supseteq (x_0, \ldots, x_n)\} \\
&= \{\underline{\lambda} \in \mathbb{A}^m \mid (F_1(\underline{X}, \lambda), \ldots) \supseteq (x_0, \ldots, x_n)^t \forall t\} \\
&= \cap_t (X_t \equiv \{\underline{\lambda} \in \mathbb{A}^m \mid W_t \equiv [F_1(\underline{x}, \underline{\lambda}), \ldots, F_r(\underline{x}, \underline{\lambda})]_{degt}) \not\supseteq V_t
\end{aligned}$$

Where $V_t \equiv$ the vector space of homogeneous degree t polynomials in $x_0, \ldots, x_n$. It is enough to show that each $X_t \subseteq \mathbb{A}^n$ is closed. W_t is obviously spanned by $\{F_i X^J\}_{|J|=degF_i, i=0,\ldots,r}$. Write each $F_i X^J$ in terms of basis X^I for $V_t \Rightarrow F_i x^J = \Sigma a_I^{i,J} x^I$, where the a_I are polynomials in over k in $\lambda_0, \ldots, \lambda_m$. The number of x^I combinations is: $\binom{n+t}{t}$. Find a matrix for this. Identify the row space with W_t. So $W_t \neq V_t \Leftrightarrow$ the rank of the matrix is less than $\binom{n+t}{t}$. $\Rightarrow X_t = \mathbb{V}(\binom{n+t}{t}$ minors of the matrix $\{a_I^{i,J}\}) \subseteq \mathbb{A}^m$ ☐

Lecture 14. Finite Maps

Algebra Blackbox: Let $R \hookrightarrow S$ be an (injective) extension of rings.

Definition 1.3.37. An element $s \in S$ is **integral over** R if it satisfies a monic polynomial with coefficients in R

Definition 1.3.38. The ring S is **integral over** R if all elements of S are integral over R

Remark 1.3.39. To check that S is integral over R, it is sufficient to check that each element in a set of algebraic generators for S/R is integral

Fact: S is a finitely generated R-module $\Rightarrow$ S is a finitely generated R-algebra and S is integral over R.

Theorem 1.3.40. ***Lying Over Theorem:*** *If $R \to S$ is integral and $P \subset R$ is a prime (or maximal) ideal then there exists a unique $q \subseteq S$ prime (or maximal) with $q \cap R = P$.*

Proposition 1.3.41. *If $V \subseteq \mathbb{P}^n$ is a projective variety and $p \in \mathbb{P}^n, p \notin V$ then the projection:*
$\pi_p : V \to H \cong \mathbb{P}^{n-1}$ is finite-to one onto its image, where H is a hyperplane not containing the point p.

Proof. Take $q \in H, \pi^{-1}(q) = V \cap L$ where $L = \overline{pq}$. $V \cap L \subset L$ is closed. The choices for $\pi^{-1}(q)$ are the empty set, L, or a finite set. Now L is not possible because if it were we would have that $L = L \cap V \Rightarrow p \in V$ which is a contradiction. ☐

Definition 1.3.42. $X \xrightarrow{\varphi} Y$ is **dominant** if $\varphi(X)$ is dense in Y.

Definition 1.3.43. A morphism of affine varieties, $X \xrightarrow{\varphi} Y$, is **finite** if it is dominant and the corresponding map of coordinate rings $k[Y] \xrightarrow{\varphi^*} k[X]$ is integral.

Definition 1.3.44. A morphism of quasi-projective varieties, $X \xrightarrow{\varphi} Y$, is **finite** if φ is dominant and $\forall y \in Y$ there is an open affine neighborhood U of y such that $\varphi^{-1}(U)$ is affine and $\varphi^{-1}(U) \xrightarrow{\varphi} U$ is finite (equivalently, $\mathcal{O}_Y(U) \xrightarrow{\varphi^*} \mathcal{O}_X(\varphi^{-1}(U))$ is integral).

Theorem 1.3.45. *If $X \xrightarrow{\varphi} Y$ is a dominant morphism of affine varieties and $\forall y \in Y \exists U$ open affine that contains y with $\varphi^{-1}(U)$ affine and $\mathcal{O}_Y(U) \xrightarrow{\varphi^*} \mathcal{O}_X(\varphi^{-1}(U))$ is integral, then also $k[Y] \xrightarrow{\varphi^*} k[X]$ is integral.*

Remark 1.3.46. This proof can be found in 5.3 of Shaf. and is similar to the proof that regular functions in an affine algebraic set is the restriction of a polynomial

Properties of Finite Maps:

Remark 1.3.47. If $X \xrightarrow{\varphi} Y$ is a finite map of quasi-projective varieties, then $\{\varphi^{-1}(q)\}$ is finite $\forall q \in Y$.

Proof. We can look at an open affine variety of Y and its inverse image, which will also be an affine variety. So, without loss of generality, we can reduce to the case when X, Y are affine varieties. Take $p \in \varphi^{-1}(q)$. Claim: there are only finitely many possibilities for each coordinate of p. $k[Y] \xrightarrow{\varphi^*} k[X] \cong k[x_1, \ldots, x_n]/\mathbb{I}(X)$ Each x_i satisfies $x_i^t + \varphi^* a_1 x_i^{t-1} + \ldots + \varphi^* a_t = 0$ where $a_i \in k[Y]$. Apply this to the point p. Note that $\varphi^* a_1(p) = a_1(\varphi(p))$. The i^{th} coordinate of p satisfies a monic polynomial with coefficients in k. At most t possibilities for the i^{th} coordinate of p. □

Remark 1.3.48. Caution: The converse of this statement is not true in general. For example consider $V = \mathbb{V}(x^2 + y^2 - 1) - \mathbb{V}(y - \frac{1}{2}) \subseteq \mathbb{A}^2$: Project this down to the x-axis. Clearly the preimage of every point is finite. The problem occurs in the coordinate rings: $k[X] \to k[x,y]/(x^2 + y^2 - 1)[\frac{1}{y-\frac{1}{2}}]$. $\frac{1}{y-\frac{1}{2}}$ is not integral over $k[X]$. So φ is not finite.

Theorem 1.3.49. *Let $X_0 \xrightarrow{\pi_0} Y_0$ be a regular map of projective varieties. Let $Y \subseteq Y_0$ be any open subset, $X = (\pi_0)^{-1}(Y)$ its inverse image, and π the restriction of π_0 to X. If the fibers of π are finite then it is a finite map.*

Proof. On page 178 of Algebraic Geometry: A First Course by Harris. □

Proposition 1.3.50. *Finite map are surjective*

Proof. Without loss of generality, we can assume that $X \xrightarrow{\varphi} Y$ is a morphism of affine varieties (again because we can look at an affine open set and its inverse image will also be affine). $k[X] \longleftarrow k[Y]$, $q = (\lambda_1, \ldots, \lambda_m)$ is given by $(y_1 - \lambda_1, \ldots, y_m - \lambda_m) = m_Q$ and $\varphi^{-1}(q)$ is defined by $(\varphi^*(y_1) - \lambda_1, \ldots, \varphi^*(y_m) - \lambda_m) \in k[X]$ (note that we want this not to be (1)). There exists a maximal ideal $M_p \in k[X]$ such that $M_p \cap k[Y] = M_p$. □

Lecture 15. Noether Normalization

Algebra Blackbox:
Let $k \hookrightarrow L$ be a field extension.

Definition 1.3.51. $x \in L$ is **algebraic over k** if it satisfies $x^n + a_1x^{n-1} + \ldots + a_n = 0$ where $a_i \in k$. Otherwise k is **transcendental**

Definition 1.3.52. Elements $x_1, \ldots, x_d \in L$ are **algebraically independent over k** if they satisfy no (non-zero) polynomial $F(u_1, \ldots, u_d) \in k[u_1, \ldots, u_d]$

Definition 1.3.53. A maximal set of algebraically independent elements of L/k is a **transcendence basis for** L/k. The cardinality of any 2 transcendence basis is the same, it is called the transcendence degree.

You can always choose a transcendence basis from any set of generators for L/k.

Basic Properties of Finite Maps (proven and yet to be proven):

(1) Finite maps are surjective with finite fibers
(2) Conversely, let X be projective and $X \xrightarrow{\varphi} Y$. If φ is a dominant map with finite fibers, then φ is finite.
(3) Compositions of finite maps are finite (since an integral extension of an integral extension is an integral extension)
(4) Finite maps are closed:

Proof. Let $X \xrightarrow{\varphi} Y$ be a finite map. It is enough to prove this for an irreducible subset $Z \subset X$. Then $Z \xrightarrow{\varphi|_Z} \overline{\varphi(Z)}$ is a finite map and since finite maps are surjective, $\varphi|_Z(Z) = \overline{\varphi(Z)}$. Therefore $\varphi(Z)$ is closed and so finite maps are closed. □

(5) Let $X \subset \mathbb{P}^n$ be projective and let $H \subset \mathbb{P}^n$ be any linear subspace. If $X \cap H = \varnothing$, then projection from H is finite. *(See Shaf. I, page 64 for a self-contained proof)*

Theorem 1.3.54 (Noether Normalization:)**.** *A projective variety V admits a finite morphism to some projective space $\mathbb{P}^d$. Moreover the d is uniquely determined.*

Let $V \subseteq \mathbb{P}^n$ be projective and irreducible. Take any point $p \in \mathbb{P}^n \setminus V$ and project from $\mathbb{P}^n$ to $\mathbb{P}^{n-1}$ through p. Let $V_1 = \pi(V)$ Then we get the following diagram:

$$\begin{array}{ccc} \mathbb{P}^n & \overset{\pi}{\underset{\text{project}}{\twoheadrightarrow}} & H \cong \mathbb{P}^{n-1} \\ \uparrow & & \uparrow \\ V & \overset{\text{project}}{\underset{\pi}{\twoheadrightarrow}} & V_1 \end{array}$$

If $V \neq \mathbb{P}^{n-1}$ then we repeat this process until it stops, which will occur when the image of V is all of $\mathbb{P}^d$ for some $d \in \mathbb{N}$. Thhis gives us a sequence: $V \to V_1 \to V_2 \to \ldots \to \mathbb{P}^d$.

A more general statement of the above theorem is:

Theorem 1.3.55. *If $R = k[x_1, \ldots, , x_n]/I$ is a domain finitely generated over k and $|k| = \infty$ there exists $y_1, \ldots, y_d$ k-linear combinations of $x_1, \ldots, x_n$ such that $y_1, \ldots, y_d$ are algebraically independent and $k[y_1, \ldots, y_d] \subseteq R$ is a finite extension.*

Corollary 1.3.56. *If V is an irreducible affine variety, then V admits a finite morphism to $\mathbb{A}^d$ where d is uniquely determined.*

2. Smoothness

2.1. Dimension of Varieties.

Definition 2.1.1. Let V be an irreducible quasi-projective variety. Define the **function field of** V, denoted $k(V)$, to be the function field of any (dense) open affine subset of V.

Remark 2.1.2. This definition is independent of the choice of an affine open set. Take any $U_1, U_2 \subseteq V$ open affine. Let $U_2 \supset U_3$ =basic open affine of $U_1 = U_1 - \mathbb{V}(g)$ for some $g \in \mathcal{O}_V(U_!)$. $\mathcal{O}_V(U_1) = k[U_1] \to k[U_3] = k[U_1][\frac{1}{g}]$.

Definition 2.1.3. The **dimension** of an irreducible quasi-projective variety V is the transcendence degree of $k(V)$ over k.

So why is d uniquely determined?

$$\pi^{-1}(U) \subseteq V \xrightarrow{\quad\pi\quad} U \subseteq \mathbb{P}^d \text{ finite and} U \text{ affine}$$

$$k[\pi^{-1}(U)] \xleftarrow[\textit{transcendence}]{} k[U] \xleftarrow[\textit{transcendence}]{} k$$

The dimension of $\mathbb{P}^d$ = transcendence degree of $k[\mathbb{P}^d]/k$.

Definition 2.1.4. The dimension of a (non-irreducible) quasi-projective variety is the maximal dimension of its irreducible components

Example 2.1.5. $\mathbb{A}^3 \supset \mathbb{V}(xz, xy) = \mathbb{V}(x) \cup \mathbb{V}(y, z)$ and so $k[y, z] \cong k[x, y, z]/k[x]$ has transcendence degree 2 and $k[x, y, z]/(y, z) \cong k[x]$ has transcendence degree 1.

Lecture 16. Dimension

Algebra Blackbox:
Definition 2.1.6. The **Krull dimension** of a commutative ring R is the length of the longest chain of prime ideals of R.

Example 2.1.7. $P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \ldots \subsetneq P_d \subsetneq R$ has length d.

Remark 2.1.8. Recall that $k(V)$ is the fraction field of the ring of regular functions $\mathcal{O}_V(U)$ on any open affine set $U \subseteq V$. If the variety is not irreducible, then just look at its (finitely many) irreducible components).

Example 2.1.9. $dim\, k(\mathbb{P}^n) = n$ and its function field is: $k(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$ then the transcendence degree is n

Basic Properties of dimension:

1. $V \cong W \Rightarrow dimV = dimW$
2. $U \subseteq V$ dense and open, then $dimV = dimU$.
3. $X \xrightarrow{\varphi} Y$ surjective and finite, then $dimX = dimY$
4. $dim(X \times Y) = dimX + dimY$

Proposition 2.1.10. *Let $X = \mathbb{V}(f) \subseteq \mathbb{A}^n$ (or $\mathbb{P}^n$) be a hypersurface. Then $dimX = n - 1$.*

Proof. If X were in $\mathbb{P}^n$ then we would look at the affine open cover of $\mathbb{P}^n$, so we can, without loss of generality, reduce to the case when $X \subseteq \mathbb{A}^n$. We can also reduce to the case when f is irreducible, since if it were not then we would look at its irreducible components. Also assume that $f \neq 0, f \neq$ constant. Let $f \in k[x_1, \ldots, x_n]$. Say that x_n appears in f. Claim: $x_1, \ldots, x_{n-1}$ are a transcendent basis for $k(X)/k$. Need to check that they are algebraically independent over k. If not, then there would exist a

$G(x_1, \ldots, x_{n-1}) = 0$ where $G \in k[x_1, \ldots, x_{n-1}]$. Then $G \in \mathbb{I}(X) = (f) \Rightarrow G = fh$, but x_n appears in f but not in G. So there cannot exist such a G. The set $\{x_1, \ldots, x_{n-1}\}$ is a maximal set of algebraically independent elements since $\{x_1, \ldots, x_n\}$ are not algebraically independent. □

Proposition 2.1.11. *Let V be irreducible, and $W \subsetneq V$ is closed, then $dimV > dimW$.*

Proof. Without loss of generality we can reduce to the affine case: $W \subsetneq V \subseteq \mathbb{A}^n$. Again we can assume that W is irreducible by the standard trick: look at its irreducible components.

$k[x_1, \ldots, x_n] \twoheadrightarrow k[V] \xrightarrow{\varphi} k[W]$. Say $dimW = dimV$. Take a transcendence basis for $k(W)$. Without loss of generality, $x_1, \ldots, x_d$ (restricted to W). Note that this is a transcendence basis for $k(V)$. Take $g \in Ker\varphi, g \neq 0$.

$$\begin{array}{ccc} k[x_1, \ldots, x_d] & \hookrightarrow & k[V] \\ \downarrow & & \downarrow \\ k(x_1, \ldots, x_d) & \xhookrightarrow{algebraic} & k(V) \end{array}$$

There exists a polynomial with coefficients in $k(x_1, \ldots, x_d)$ satisfied by g: Then $a_0 g^T + a_1 g^{T-1} + \ldots + a_T = 0, a_i \in k[x_0, \ldots, x_n]$. This holds in $k[V]$.
Restrict to W. $a_T(x_1, \ldots, x_d) = 0$ in $k[W]$. This is a contradiction. □

Theorem 2.1.12. *Let V be a quasi-projective variety. Then $dimV$ is equal to the length of the longest chain of closed irreducible subvarieties of V.*

For instance: $V \supseteq V_d \supsetneq V_{d-1} \supsetneq \ldots \supsetneq V_1 \supsetneq V_0 = \{point\}$.

Remark 2.1.13. The length of the longest chain of closed irreducible subvarieties of V is the Krull dimension.

Example 2.1.14. $\mathbb{A}^d \supsetneq \mathbb{V}(x_1) = \mathbb{A}^{d-1} \supsetneq \mathbb{V}(x_1, x_2) \supsetneq \ldots \supsetneq \mathbb{V}(x_1, \ldots, x_{d-1}) \supsetneq \mathbb{V}(x_1, \ldots, x_d) = 0$. This is a chain of length d.

Proof. First direction: $dimV \geq$ Krull dimension of V. Since $V \supseteq V_d \supsetneq V_{d-1} \not\supseteq \ldots \not\supseteq V_1 \not\supseteq V_0 = \{point\}$, it is immediate from a previous proposition that a proper closed set has strictly smaller dimension.
Second direction: Say V has dimension d. Without loss of generality, V is irreducible, projective so that $V \supseteq \overline{V} \subseteq \mathbb{P}^n$. □

Lemma 2.1.15. *Given any projective variety, $V \subset \mathbb{P}^n$, there is a hyperplane $H \subseteq \mathbb{P}^n$ not containing any components of V.*

Proof. $V \supsetneq V \cap H_0 \supsetneq V \cap H_0 \cap H_1 \supsetneq \ldots \supsetneq V \cap H_0 \cap H_1 \cap \ldots \cap H_T = \varnothing$
This is a chain of length T of irreducible, closed sets of V. Note that $H_1 \cap \ldots \cap H_T$ is a linear space.
We need to show that $T \geq d$. Say that $H_i = \mathbb{V}(L_i), L_i$ is some linear polynomial.
Let $\Lambda = \mathbb{V}(L_0, \ldots, L_T) = H_0 \cap \ldots \cap H_T$, this is a linear space. Project from Λ:

$$V \xrightarrow{\pi_\Lambda} \mathbb{P}^T$$

where $x \mapsto [L_0(x) : \ldots : L_T(x)]$. Projections are finite and surjective onto its image, so $V \underset{\pi_\Lambda}{\overset{finite}{\twoheadrightarrow}} \pi_\Lambda(V) \subseteq \mathbb{P}^T$ and $dimV = dim\pi_\Lambda(V) \leq T$. □

Lemma 2.1.16. *(More general) Given any projective variety $V \subset \mathbb{P}^n, M \in \mathbb{N}, \exists f_M$ homogeneous polynomial of degree M such that $\mathbb{V}(f_M)$ does not contain any component of V.*

Proof. Choose p_i on component V_i where $V = V_1 \cup \ldots \cup V_r$. We need to find f_M such that $P_i \notin \mathbb{V}(f_M)$. But the set

$$H_{p^i} \subseteq \mathbb{P}(Sym^M(V^*))$$

of hypersurfaces containing P_i i a hypersurface in $\mathbb{P}(Sym^*(V^*))$. So, $U_{p_1 \ldots p_r} = U_{p_1} \cap U_{p_2} \cap \ldots \cap U_q$ open, dense in $\mathbb{P}(Sym^M(V^*))$.

$$U = \bigcup_{p_i \in V_i} U_{p_1 \ldots p_r} \subseteq \mathbb{P}(Sym^M(V^*))$$

□

Lecture 17.

Algebra Blackbox:

Remark 2.1.17. If R is a domain and $f \in R - \{0\}$, then all minimal primes of (f) have height 1. In general, the minimal primes of $(f_1, \ldots, f_r)$ have height $\leq r$.

Corollary 2.1.18. *If V is an irreducible affine (or projective) variety of dimension d, and f is a (homogeneous) polynomial not vanishing on V, then dim $(V \cap \mathbb{V}(f)) = d - 1$ and all components are also of dimension $d - 1$.*

Remark 2.1.19. In particular, hyperplane sections of V have dimension one less than V, $V \neq V \cap H$.

Example 2.1.20. Let $V = \mathbb{V}(x^2 + y^2 z^2) \subseteq \mathbb{P}^3$. One hyperplane section is $V \cap \mathbb{V}(z - 4)$.

Corollary 2.1.21. *Let V be a codimension 1 subvariety of $\mathbb{A}^n$ (or $\mathbb{P}^n$). Then $V = \mathbb{V}(f)$ for some (homogeneous) polynomial in n (or n+1) variables.*

Proof. Let's assume V is affine. Without loss of generality, V and f are irreducible, where $f \in \mathbb{I}(V) \subseteq k[x_1, \ldots, x_n]$ prime ideal. $(f) \subseteq \mathbb{I}(V) \Rightarrow \mathbb{V}(f) \supseteq \mathbb{V}(\mathbb{I}(V)) = V$. Now $dim\mathbb{V}(f) = n - 1$ and $dimV = n - 1 \Rightarrow \mathbb{V}(f) = V$. □

Corollary 2.1.22. *A quasi-projective variety of dimension d contains subvarieties of every smaller dimension.*

Proof. Unless you are very unlucky, any general hyperplane will cut a variety of dimension d into one of dimension d-1. Repeat this process. □

Corollary 2.1.23. *If V is projective of dimension n and $F_1, \ldots, F_r$ are homogeneous polynomials,*
then dim$(V \cap \mathbb{V}(F_1, \ldots, F_r)) \geq n - r$ and if the F_i are "sufficiently general" then we have equality, where negative $n - r$ means the variety is empty. In particular, a closed subvariety of $\mathbb{P}^n$ defined by r homogeneous equations has dimension $\geq n - r$.

Remark 2.1.24. "Sufficiently general" or "generic" means that there is an unspecified open subset of the variety in which you are looking that has this property.

Example 2.1.25. In $\mathbb{P}^2$ intersects, every 2 curve intersects, however this is false on general varieties. For instance, consider $\{q\} \times \mathbb{P}^1, \{p\} \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$.

Definition 2.1.26. A projective variety $V \subseteq \mathbb{P}^n$ is a **set-theoretic complete intersection** if it has codimension r and it is defined by r homogeneous equations.

Open question: Is every curve in $\mathbb{P}^3$ an intersection of 2 surfaces (i.e. a complete intersection)?

Definition 2.1.27. A projective variety $V \subseteq \mathbb{P}^n$ is a **scheme-theoretic complete intersection** if
$\mathbb{I}(V) \subseteq k[x_0, \ldots, x_n]$ is generated by codimension V polynomials.

Remark 2.1.28. The set-theoretic definition is much weaker than the scheme theoretic definition.

Let $X \xrightarrow{\varphi} Y$ where both are irreducible varieties.
What can be said about the relationship between $dimX, dimY$, and $dim\{\varphi^{-1}(y)\}_{y\in Y}$?
For generic $y \in Y, dimY + dim\varphi^{-1}(y) = dimX$. This is not true for all $y \in Y$.

Theorem 2.1.29. ***Dimension of Fibers:*** *If $X \xrightarrow{\varphi} Y$ is a surjective map of irreducible varieties, then:*

1. *$dimX \geq dimY$*
2. *$dim\{\varphi^{-1}(y)\} \geq dimX - dimY$*
3. *There's a non-empty open set $U \subseteq Y$ such that $\forall y \in U$, equality holds in (2).*
4. *The set $\{y \in Y \mid dim\varphi^{-1}(y) = dimX - dimY\}$ is open and non-empty in Y*
5. *The set $Y_l = \{y \in Y \mid dim\varphi^{-1}(y) \geq l\}$ is closed in Y.*

Proof. (Proof of 1) Reduce to the case of affine (dominant):
$k[Y] \xhookrightarrow{\varphi^*} k[X]$ and $k[Y] \subseteq k(Y) \hookrightarrow k(X) \Rightarrow$ transcendence degree of $k(X)/k \geq k(Y)/k$. Note that the transcendence degree of $k(X)/k(Y)$ is n-m.
(Proof of 2):
$\{\varphi^{-1}(y)\} = \mathbb{V}(\varphi^* f_1, \ldots, \varphi^* f_d) \subseteq \varphi^{-1}(U) \xrightarrow{\varphi} U \subseteq Y$, where U is an open neighborhood of y. And $dim\mathbb{V}(\varphi^* f_1, \ldots, \varphi^* f_d) \geq dimX - d$. □

Lecture 18.

Remark 2.1.30. If $X \xrightarrow{\varphi} Y$ is a finite map of irreducible varieties, then $dimX = dim($ any generic fiber $)+ dimY$

Example 2.1.31. The canonical line bundle: the general fiber is of dimension 0, but $\pi^{-1}((0,0)) \cong \mathbb{P}^1$.

Lemma 2.1.32. *Given $y \in Y$ irreducible of dimension d. Then $\exists F_1, \ldots, F_d$ regular functions on some neighborhood U of y such that $\mathbb{V}(F_1, \ldots, F_d) \cap U = \{y\}$.*

Sketch of Proof. You can choose regular functions F_i such that $dim(V \cap \mathbb{V}(F_1, \ldots, F_d)) = 0$, which only contains points. Since points are closed, we can easily choose an open neighborhood of our point y that does not contain any of these points. □

Corollary 2.1.33. *Given that $X \xrightarrow{\varphi} Y$ is a surjective morphism of projective varieties, Y is irreducible and the fibers $\varphi^{-1}(y)$ are all irreducible of the same dimension. Then X is irreducible (of dimension $dimY+$ fiber dimension).*

Proof. See Shaf. 6.2 □

Remark 2.1.34. If X and Y are irreducible projective varieties, then $X \times Y$ is irreducible. One can proof this by defining $X \times Y \xrightarrow{\varphi} Y$ and using the preceding corollary.

When does a hypersurface $X \subseteq \mathbb{P}^n$ of degree d contain a line?
Clearly true when d is 1, but when d gets larger and larger, the hypersurface can get more and more mangled.
$\Sigma = \{(L, X) \mid L \subseteq X\} \subseteq Gr(2, V) \times \mathbb{P}(Sym^d(V^*)) \cong$ the set of lines in $\mathbb{P}(v)$ times $\mathbb{P}^{\binom{d+n}{n}-1}$.

Let $G = Gr(2, V)$ and $m = \binom{d+n}{n} - 1$. So Σ is a closed subset of $G \times \mathbb{P}^m$, hence it is a projective variety.

Consider the projection: $\Sigma \xrightarrow{\pi} G$. Compute $\pi^{-1}(L)$:
π is surjective. Choose coordinates so that $L = \{[a : b : 0 : \ldots : 0]\} = \mathbb{V}(x_2, \ldots, x_n)$.
Let $F = \Sigma a_I x^I$. Then if:

$$\begin{aligned} X = \mathbb{V}(F) \supseteq L &\Leftrightarrow \mathbb{I}(X) = (F) \subseteq \mathbb{I}(L), \text{ that is } (F) \subseteq (x_2, \ldots, x_n) \\ &\Leftrightarrow \text{there is no } x_0^d, x_0^{d-1}x_1, \ldots, x_n^d \text{ term appearing in } F \\ &\Leftrightarrow a_I = 0 \text{ when } I \text{ is a coefficient only in } x_0, x_1 \end{aligned}$$

So $\pi^{-1}(L) = \mathbb{V}(a_{(d,0,\ldots,0)}, \ldots, a_{(0,d,0,\ldots,0)}) \subseteq \mathbb{P}(Sym^d(V^*)) = \mathbb{P}^M$. That is, fibers over any L are irreducible and of the same dimension. So Σ is irreducible, of dimension $M - (d+1) + 2(N-1) = 2N\binom{d+N}{N} - d - 4$.

Consider the projection $\Sigma \xrightarrow{\pi_2} \mathbb{P}(Sym^d(V^*))$. The image of this map is the closed set of hypersurfaces containing a line! If $dim\Sigma < dim\mathbb{P}(Sym^d(v^*)), \pi_2$ is NOT surjective so the general hypersurface of degree d contains NO line.
Σ is irreducible of dimension: $m - (d+1) + 2(n-1)$. So for large d, we have NO line on the general hypersurface!

Example 2.1.35. Case of $\mathbb{P}^3$: Σ is of dimension $m + 3 - d$. If $d > 3$, then $dim\Sigma < m$ and there are no lines on the general degree d surface in $\mathbb{P}^3$. If $d = 1, X \cong \mathbb{P}^2$ which has lots of lines. If $d = 2, X = \mathbb{V}(xy - wz) \subseteq \mathbb{P}^?$ which is a 1-dimensional family of lines. If $d = 3, \Sigma$ has dimension m (which is 19). The surface $\mathbb{V}(x_0^3 - x_1x_2x_3) \subseteq \mathbb{P}^3$ contains only finitely many lines. Note that $\mathbb{V}(x_0^3 - x_1x_2x_3) \cap \mathbb{V}(x_0)$ has 3 lines.

Remark 2.1.36. All cubic surfaces in $\mathbb{P}^3$ contain at least 1 line and the general cubic surface contains finitely many lines.

2.2. **Tangent Spaces.**

Lecture 19. "Extrinsic" Approach to Tangent Spaces

Goal: Determine the tangent space to a variety V at a point $p \in V$.

Example 2.2.1. $p = (0,0) \in V = \mathbb{V}(y - x^2) \subseteq \mathbb{A}^2$ Let $l = \{(ta, tb) \mid t \in k\}$. Then consider $V \cap l$:
Solve for $t \Rightarrow t(b - a^2t) = 0 \Rightarrow t = 0$ or $t = \frac{b}{a^2}$. We get two distinct points unless $b = 0$

Idea: Tangency is a local condition, near p, so we can assume that $V \subseteq \mathbb{A}^n$ is a Zariski-closed set and $p = \underline{0} \in V$. Then $T_pV \subseteq \mathbb{A}^n$: will consist of all points on all lines tangent to V at p.

We need to address the following concerns:

(1) When is a line L tangent to V at p?
(2) Why is it a vector space?
(3) Why is it intrinsic to p?

Definition 2.2.2. Let $p = 0 \in V \subseteq \mathbb{A}^n$ with $\mathbb{I}(V) = (F_1, \ldots, F_m)$ and a line L such that $p \in L = \{t(a_1, \ldots, a_n) = t\underline{a} \mid \underline{a} \neq 0\}$. The **intersection multiplicity of** $L \cap V$ **at** p is the highest power of t dividing the $gcd(F_1(t\underline{a}), \ldots, F_m(t\underline{a})) \in k[t]$.

Definition 2.2.3. The line L is **tangent to** V **at** p if the intersection multiplicity of L and V at p, $(L \cdot V)_p$, is greater than or equal to two.

Definition 2.2.4. The **tangent space to** V **at** p, denoted T_pV, is the set of all points lie on a line tangent to V at p. In particular,

$$T_pV = \{(x_1, \dots, x_n) \in \mathbb{A}^n \mid (x_1, \dots, x_n) \in L, \text{ where } L \text{ is a line tangent to } V \text{ at } p\}$$

Let's analyze these definitions:
Consider $p = \underline{0}$ where $p \in V = \mathbb{V}(F_1, \dots, F_m)$. Then $F_i(p) = 0$ and we can write each F_i as the sum of a linear polynomial, L_i, and a polynomial with higher order terms, G_i so that $F_i = L_i + G_i$.

$$F_i(ta_1, \dots, ta_n) = tL_i(a_1, \dots, a_n) + t^2 H_i(t), \text{ for some polynomial such that } t^2 H_i(t) = G_i(t).$$

The line L is tangent to V at p if and only if t^2 divides $F_i(t\underline{a}) \forall i$ if and only if $L_i(\underline{a}) = 0 \; \forall \; i$.
Therefore $T_pV = \mathbb{V}(L_1, \dots, L_m) \subseteq \mathbb{A}^n$ where L_i is the linear part of F_i.

Remark 2.2.5. The tangent space is the linear variety most closely approximating V at the point p.

Definition 2.2.6. The **differential of** F **at** $p = (\lambda_1, \dots, \lambda_n)$, denoted by d_pF, is : $\Sigma_{j=1}^n \frac{\partial F_i}{\partial x_j}|_p (x_j - \lambda_j)$.

Now consider $p = (\lambda_1, \dots, \lambda_n)$ where $p \in \mathbb{V}(F_1, \dots, F_m)$. We can write each F_i in terms of L_i, linear polynomials in $(x_1 - \lambda_1), \dots, (x_n - \lambda_n)$, and G_i, polynomials of degree at least 2 in $(x_1 - \lambda_1), \dots, (x_n - \lambda_n)$.
We can approximate the F_i by using their Taylor expansions:

$$F_i = \Sigma_{j=1}^n \frac{\partial F_i}{\partial x_j}|_p (x_j - \lambda_j) + \text{ higher-order terms in } (x_1 - \lambda_1), \dots, (x_n - \lambda_n)$$

Hence, $T_pV = \mathbb{V}(d_pF_1, \dots, d_pF_m) \subseteq \mathbb{A}^m$, where the F_i's generate $\mathbb{I}(V)$.

Example 2.2.7. Let $p = \underline{0} \in V = \mathbb{V}(y - x^2), d_p(y - x^2) = 2x|_p(x - 0) + 1|_p(y - 0) = y \Rightarrow T_pV = \mathbb{V}(y)$.

Example 2.2.8. Let $p = (0, 0, 1) \in \mathbb{V}(x^2 + y^2 + z^2 - 1) \subseteq \mathbb{A}^3$.
$T_pV = \mathbb{V}(d_p(x^2 + y^2 + z^2 - 1)) = \mathbb{V}(2z|_p(z - 1)) = \mathbb{V}(z - 1)$, if the characteristic is not 2.

Example 2.2.9. Let $p = (0, 0, 0) \in \mathbb{V}(x^2 + y^2 - z^2) \subseteq \mathbb{A}^3$
$T_pV = \mathbb{V}(d_p(x^2 + y^2 - z^2)) = \mathbb{V}(0) = \mathbb{A}^3$. since p is a singularity.

Properties of Differentials:
Let $F \in k[x_1, \dots, x_n]$ and $p = (\lambda_1, \dots, \lambda_n)$

1. $d_pF = \Sigma_{i=1}^n \frac{\partial F}{\partial x_i}|_p (x_i - \lambda_i)$
2. $d_p(F + G) = d_pF + d_pG$
3. $d_p(\lambda F) = \lambda d_pF$, where $\lambda \in \mathbb{R}$
4. Leibniz Rule: $d_p(FG) = F(p) d_pG + G(p) d_pF$

Lecture 20. "Intrinsic" Approach to Tangent Spaces

Concern: What if we have an isomorphism between varieties, $V \xrightarrow[\cong]{\varphi} W$ sending $p \mapsto q$. Is there an isomorphism of varieties, $T_pV \xrightarrow[\cong]{T_p(\varphi)} T_qW$, that preserves the vector space structure?

Goal: We want a more intrinsic way to define and think about tangent space that depends only on $p \in V$

The map $k[x_1, \dots, x_n] \xrightarrow{d_p} \{$ linear polynomials taking $p = (\lambda_1, \dots, \lambda_n) \to 0\}$ is k-linear.
This map descends to a map on coordinate rings:
$k[V] \xrightarrow{d_p} (T_pV)^*$ mapping $f = F|_V \mapsto d_pf = d_pF|_{T_pV}$.

We need to check that d_p is well-defined. It suffices to show that if $F|_V = 0$, then $d_pF|_{T_pV} = 0$.
Say $F \in \mathbb{I}(V) \Rightarrow F = H_1F_1 + \ldots + H_mF_m$. Note that $f_i(p) = 0$
$d_pF = \Sigma_{i=1}^n d_p(H_iF_i) = \Sigma_{i=1}^n H_i(p)d_pF_i + F_i(p)d_pH_i = 0$ by Leibniz rule. Therefore this map is well-defined.
Restrict to $m_p =$ functions regular at $p \subseteq k[V]$.
Note that $d_p(f) = d_p(f - f(p))$ since $f(p) = 0$.
Let $m_p \longrightarrow (T_pV)*$
Claim: This is a surjective k-vector space morphism with kernel m_p^2.
Therefore, d_p induces an isomorphism $m_p/m_p^2 \xrightarrow{\cong} (T_pV)^*$. So m_p/m_p^2 is intrinsic vector space associated to $p \in V$, isomorphic to co-tangent space $(T_pV)^*$.

Proof. (Of Claim) m_p is the maximal ideal of regular functions vanishing at p, so $m_p = (x_1 - \lambda_1, \ldots, x_n - \lambda_n)$. To see that this map is surjective, $(T_pV)^*$ is spanned by the restrictions of $x_i - \lambda_i$. $d_p(x_i - \lambda_i) = x_i - \lambda_i \Rightarrow$ surjective.
Check that $m_p^2 \subseteq kerd_p$:
$d_p(fg) = f(p)d_pg + g(p)d_pf = 0$ since $f(p) = g(p) = 0$.
Now check that $m_p^2 \supseteq kerd_p$:
Take $g = G|_V \in kerd_p$. $G = G(p) + \Sigma\frac{\partial G}{\partial x_i}(x_i - \lambda_i) + \Sigma\frac{\partial^2 G}{\partial_i \partial x_j}(x_i - \lambda_i)(x_j - \lambda_j)+$ higher-order terms.
$G(p) = 0, \Sigma\frac{\partial G}{\partial x_i}(x_i - \lambda_i) = d_pG = 0$ since we took $G \in kerd_p$, and $\Sigma\frac{\partial^2 G}{\partial_i \partial x_j}(x_i - \lambda_i)(x_j - \lambda_j) \in m_p^2 \Rightarrow$
$G|_V = g \in m_p^2$. Therefore $kerd_p = m_p^2$. ☐

Definition 2.2.10. The **Zariski tangent space** to a point p on a quasi-projective variety V will be defined $(m_p/m_p^2)^*$, where $m_p \subseteq \mathcal{O}_{V,p}$ is the maximal ideal of regular functions vanishing at p.

Recall: If $p \in V$quasi-projective variety then we can define

$$\begin{aligned}
\mathcal{O}_{V,p} &= \text{ the local ring of } V \text{ at } p \\
&= \text{all functions that are regular at } p \\
&= \text{ the ring of regular functions on some unspecified neighborhood } U \text{ of } p \\
&= \{\varphi : U \to k \mid \varphi \text{ regular functions on open } U \ni p\}/[(\varphi : U \to k) \sim (\varphi' : U' \to k) \text{ if} \varphi|_W = \varphi'|_W \text{ for } W \subseteq U \cap U \\
&= \varinjlim_{p \in U} \mathcal{O}_V(U) = \{\frac{f}{g} \mid f, g \in k[U] = \mathcal{O}_V(U) \text{ for some affine } U \ni p, g(p) \neq 0\} \\
&= \{\frac{f}{g} \mid f, g \in k[U], g \notin m_p\} \\
&= k[U]_{m_p}
\end{aligned}$$

Example 2.2.11. $\mathbb{P} = (0, 0) \in U_f = \mathbb{A}^2 - \mathbb{V}(f) \subseteq \mathbb{A}^2$ such that $f(p) \neq 0.\mathcal{O}_{\mathbb{A}^2}(U_f) = k[x, y, \frac{1}{f}]$. So, $\mathcal{O}_{\mathbb{A}^2}(U_{(x-1)}) = k[x, y, \frac{1}{x-1}]$ and $\mathcal{O}_{\mathbb{A}^2}(U_{(x-1)(x-2)}) = k[x, y, \frac{1}{x-1}, \frac{1}{y-1}]$

Let $V \xrightarrow{\varphi} W$ be a morphism of varieties that sends $p \mapsto q$.
φ induces a k-vector space map: $T_pV \xrightarrow{d_p\varphi} T_qW$.
Assume, without loss of generality, that V and W are affine. Then we get the pull-back: $k[V] \xleftarrow{\varphi^*} k[W]$
Note that the pull-back satisfies the conditions: $\varphi^*(m_q^2) \subseteq m_p^2$ and $\varphi^*(m_q) \subseteq m_p$. This gives us an induced map: $m_p/m_p^2 \xleftarrow{\overline{\varphi^*}} m_q/m_q^2$

By dualizing we get: $(m_p/m_p^2)^* \xrightarrow{d_p\varphi} (m_q/m_q^2)^*$

Suppose that $\mathbb{A}^n \xrightarrow{(F_1,\dots,F_m)} \mathbb{A}^m$, where $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$.
Suppose $V \xrightarrow{\varphi} W$ by sending $p \mapsto q$. Then $T_pV \xrightarrow{d_p\varphi} T_qW$, where $d_p\varphi = \left(\frac{\partial F_i}{\partial x_j}|_p\right)_{i,j}$.

2.3. **Smoothness.**

Lecture 21.

Remark 2.3.1. The Zariski tangent space is finite dimensional.

Definition 2.3.2. The **embedding dimension** of V at p is the dimension of m/m^2.

Remark 2.3.3. If there is an open neighborhood of $p \in V$ that is isomorphic to a closed subset of $\mathbb{A}^n$, then $dimT_pV \leq n$.

Remark 2.3.4. We can do this much more generally (in terms of schemes)! $p \in SpecR, R_p =$ local ring at p. Then $T_p(SpecR) = (PR_p/(PR_p)^2)^*$ is the Zariski tangent space (dual over $R_p/(PR_p)$).

Definition 2.3.5. A point p on a quasi-projective variety V is a **smooth** point of V if $dimT_pV = dim_pV$. Otherwise, p is a **singular point.**

Example 2.3.6. $\mathbb{V}(xy, xz) \subseteq \mathbb{A}^3$: is a line ($\mathbb{V}(y,z)$) sticking out of the yz-plane ($\mathbb{V}(x)$). A point p on $\mathbb{V}(y,z)$ has $dimT_pV = 1 = dim_pV$. A point q on $\mathbb{V}(x) - \mathbb{V}(y,z)$ has $dimT_qV = 2 = dim_qV$. The point in the intersection, 0, has $dim_0V = 2$, while $dimT_0V = 3$.

Example 2.3.7. Let $V = \mathbb{V}(f) \subseteq \mathbb{A}^n$, without loss of generality assume that f has no repeated factors and is nonconstant. $\mathbb{I}(V) = Rad(f) = (f)$. For any point $p = (\lambda_1, \dots, \lambda_n) \in V, dim_pV = n-1$. $T_pV = \mathbb{V}(d_pf) = \mathbb{V}(\Sigma_i \frac{\partial f}{\partial x_i}|_p(x_i - \lambda_i)) \subseteq \mathbb{A}^n$. $dimT_pV \geq n-1$ with equality unless all the partial derivatives vanish. Singularities of $V = SingV = V \cap \mathbb{V}(\{\frac{\partial f}{\partial x_i}\})$, which is closed! $SingV$ is a proper closed set because not all $\frac{\partial f}{\partial x_i}$ can vanish on V. If $\frac{\partial f}{\partial x_i}$ vanishes on V, $\frac{\partial f}{\partial x_i} \in \mathbb{I}(V) = (f)$. Let $f = a_m x_i^m + a_{m-1}x_i^{m-1} + \dots + a_0, a_i \in k[x_1, \dots, \hat{x_i}, \dots, x_n]$. Note that the degree of $\frac{\partial f}{\partial x_i}$ in $x_i \leq$ the degree of f in x_i. This is a contradiction unless all of the $\frac{\partial f}{\partial x_i}$ are zero polynomials. In particular, $f = \Sigma a_I x_{Ip} = (\Sigma (a_i)^{\frac{1}{p}} x^I)^p)$, which is contrary to our assumption that f has no repeated roots. Therefore $SingV$ is proper.

Remark 2.3.8. In characteristic p, we prefer that our fields be algebraically closed and perfect (i.e. every element has a p-th root)

Theorem 2.3.9. *Every irreducible quasi-projective variety V has a (non-empty) open affine subset isomorphic to a hypersurface in some $\mathbb{A}^n$*

Lemma 2.3.10. *Let V be a quasi-projective variety with irreducible components $V_1, \dots, v_t$ so that $V = V_1 \cup \dots \cup V_t$. Then $SingV = SingV_1 \cup \dots SingV_t \cup (\cup_{i,j}(V_i \cap V_j)))$.*

Theorem 2.3.11. *Let V be a quasi-projective variety. Then the locus of smooth points $p \in V$ is a dense open set. Its complement, $SingV$, consists of points p, where $dim_pV < dimT_pV$.*

Proof. By the preceding lemma, we can reduce to the case where V is irreducible. Furthermore, we can reduce to the affine case by taking an affine open cover of V. Take a point $p = (\lambda_1, \dots, \lambda_n) \in V \subseteq \mathbb{A}^n$, where V is closed. For some $F_1, \dots, F_m \in k[x_1, \dots, x_n], \mathbb{I}(V) = (F_1, \dots, F_m)$. Then $T_pV = \mathbb{V}(d_pF_1, \dots, d_pF_m) = \Sigma_{j=1}^n (\frac{\partial F_i}{\partial x_j}|_p(x_j -$

$x_i)) \subseteq \mathbb{A}^n$. We can express T_pV as the common zero set of the m equations given by the rows of the following matrix:

$$\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} x_1 - \lambda_1 \\ \vdots \\ x_n - \lambda_n \end{pmatrix}$$

The dimension of the linear space $T_pV = n - rank(\big(\frac{\partial F_i}{\partial x_j}|_p\big)_{i,j})$. Define H_r as follows:

$$\begin{aligned} H_r &\equiv \{p \in V \mid dimT_pV \geq r\} \\ &= \{p \in V \mid codimT_pV \leq n - r\} \\ &= \{p \in V \mid rank(\big(\frac{\partial F_i}{\partial x_j}|_p\big)_{i,j}) \leq n - r\} \\ &= \mathbb{V}\big((n - r + 1)\text{-minors of } \big(\frac{\partial F_i}{\partial x_j}\big)_{i,j}\big) \cap V \end{aligned}$$

So H_r is a closed set of V and $H_{d+1} \subseteq H_d \subseteq H_{d-1} \subseteq \ldots \subseteq V$, where d is the dimension of V. Therefore $H_d = H_i$ for all $i < d$. By the preceding theorem, there is a dense set of points $p \in V$ such that $dimT_pV = dimV$. So H_d is dense in V and since H_d is closed, $H_d = V$. □

Example 2.3.12. An irreducible variety can have singular points. Consider $\mathbb{V}(x^2+y^2-z^2) \subseteq \mathbb{A}^3$.

Remark 2.3.13. The dim_pV is never greater than $dimT_pV$.

Lecture 22. Rational Maps and Birational Equivalence

Let $k \hookrightarrow K$ be a finitely generated field extension, $k = \overline{k}$ (or k perfect). Then there exists a transcendence basis $x_1, \ldots, x_d$ for K/k such that $k(x_1, \ldots, x_d) \hookrightarrow K$ is separable (a separating transcendent basis).
If $L \subseteq K$ is a finite separable extension of fields, then $K \cong L(\theta) \cong L[Y]$ for some $\theta \in K$ and where Y satisfies the equation $Y^n + a_1Y^{n-1} + \ldots + a_n$ for $a_i \in L$.

Definition 2.3.14. A **rational map** $V \overset{\varphi}{\dashrightarrow} W$ is a regular map on some (unspecified) dense open subset U of V such that $U \xrightarrow{\varphi} W$.

Example 2.3.15. $\mathbb{A}^2 \dashrightarrow \mathbb{A}^2$ sending $(x, y) \mapsto (\frac{1}{x}, \frac{1}{y})$ is regular on $U = \mathbb{A}^2 - \mathbb{V}(x, y)$.

Definition 2.3.16. A **rational map of quasi-projective varieties** $V \overset{F}{\dashrightarrow} W$ is an equivalence class of regular maps $\{U \xrightarrow{\varphi} W \mid U \subseteq V \text{ dense open}\}$ where the equivalence relation is defined by $\{U \xrightarrow{\varphi} W\} \sim \{U' \xrightarrow{\varphi'} W\}$ if $\varphi|_{U\cap U'} = \varphi'|_{U\cap U'}$.

Example 2.3.17. Projection from $p = [1 : 0 : \ldots : 0]$, where $\mathbb{P}^n \overset{\pi}{\dashrightarrow} \mathbb{P}^{n-1}$ by $[x_0 : \ldots : x_n] \mapsto [\frac{x_1}{x_0} : \ldots : \frac{x_n}{x_0}]$. As we have since this map is regular on $\mathbb{P}^n \setminus \{p\}$ and it is a rational map.

Definition 2.3.18. Given a rational map $V \overset{F}{\dashrightarrow} W$, the **locus of indeterminacy** of F is the set of points at which F is not regular (i.e. undefined).

Remark 2.3.19. The set of points where F is regular (i.e. defined) is open and it's complement, the locus of indeterminacy of F, is necessarily a proper, closed set.

CAUTION: Rational maps are not actually maps because they are not defined at every point, so one needs to be careful when restricting or composing them. For instance one cannot restrict a rational map to something in its locus of indeterminacy. Rational maps can be composed if the image of the first one is dense.

Remark 2.3.20. Let $V \xrightarrow{\varphi} W$ be a regular map between quasi-projective varieties. If we think of V and W as subsets of $\mathbb{P}^n$ and $\mathbb{P}^m$, respectively, then $\varphi : x \mapsto [\varphi_0(x) : \ldots : \varphi_m(x)]$, where each $\varphi_i \in \mathcal{O}_V(U)$ for some open, dense $U \subseteq V$. We can write each φ_i as $\frac{F_i}{G_i}, F_i, G_i$ where they have the same degree. So we can think of φ as $[H_0(x) : \ldots : H_m(x)]$, where the H_i are all homogeneous polynomials of the same degree. Hence, every rational map of quasi-projective varieties can be expressed this way. We do not have to worry if all the H_i share a common zero, because then that point will be contained in the locus of indeterminacy, a proper, closed set.

Definition 2.3.21. Irreducible varieties V and W are **birationally equivalent**, denoted $V \sim W$, if there are dominant rational maps $V \overset{F}{\dashrightarrow} W$ and $W \overset{G}{\dashrightarrow} V$ such that $F \circ G$ and $G \circ F$ are the identity rational maps on W and V, respectively.

Example 2.3.22. Let $\mathbb{A}^2 \overset{F}{\hookrightarrow} \mathbb{P}^2$ and $\mathbb{P}^2 \overset{G}{\hookrightarrow} \mathbb{A}^2$ be given by $F(t_1, t_2) = [1 : t_1 : t_2]$ and $G([x_0 : x_1 : x_2]) = (\frac{x_1}{x_0}, \frac{x_2}{x_0})$. $F \circ G = Id$ and $G \circ F = Id$, so $\mathbb{A}^2 \sim \mathbb{P}^2$.

Example 2.3.23. If $U \subseteq V$ both irreducible, U open, nonempty and dense $\Rightarrow U \sim V$.

Proposition 2.3.24. *Fix V, W irreducible varieties. The following are equivalent:*

1. $V \sim W$
2. $\exists U \subseteq V, U' \subseteq W$ *both open and dense, with* $U \cong U'$
3. $k(V) \cong k(W)$ *as extensions of* k.

Proof. $1 \Rightarrow 2 \Rightarrow 3$ is clear.
($3 \Rightarrow 1$): Without loss of generality, $V \subseteq \mathbb{A}^n$ affine. $k[V] \cong k[x_1, \ldots, x_n]/\mathbb{I}(V)$ and $k[W] \cong k[y_1, \ldots, y_m]/\mathbb{I}(W)$. $k(W) \xrightarrow{\cong} k(V)$ by $g_i \mapsto \frac{f_i}{g_i}$, where $f_i, g_i \in k[V], g_i \neq 0$.
We can now define a map from V to W: $(x_1, \ldots, x_n) \in V \longrightarrow (\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_m(x)}{g_m(x)}) \in W$.
This is a rational map. □

Theorem 2.3.25. *Every irreducible quasi-projective variety V contains a dense open set, U, with $U \cong \mathbb{V}(G) \subseteq \mathbb{A}^{n+1}$ for some polynomial $G \in k[x_1, \ldots, x_n]$.*

Proof. $K = k(V) \supseteq k(x_1, \ldots, x_n)$ is separable. $K = L[Y]/(y^d + a_1 y^{d-1} + \ldots + a_d)$, where $a_i \in L$.
We can convert $y^d + a_1 y^{d-1} + \ldots + a_d$ to $G \equiv b_0 y^d + b_1 y^{d-1} + \ldots + b_d$ where $b_i \in k[x_1, \ldots, x_d]$.
So $K =$ fraction field of $\big(k[x_1, \ldots, x_d, y]/(b_0 y^d + \ldots + b_d)\big)$ □

2.4. **Desingularizing Varieties.**

Lecture 23. Smoothness in Families

Definition 2.4.1. A **family of varieties** is a surjective morphism $X \xrightarrow{\pi} B$ of varieties. The fibers are the **members** of the family. The base, B, **parametrizes** the members of the family, $\{\pi^{-1}(b)\}_{b \in B}$

Example 2.4.2. $X = \mathbb{V}(xy - z) \subseteq \mathbb{A}^3$, where $\pi(x, y, \lambda) = \lambda$, gives the hyperbola family (equivalently the family is: $\{\mathbb{V}(xy - \lambda) \subseteq \mathbb{A}^2\}_{\lambda \in \mathbb{A}^1}$. Note that this family could also be describe by this map: $(x, y) \mapsto xy$.

General Principle:

(1) Nearly every "family" (in the loose sense) that we encounter in algebraic geometry is a family in the technical sense.
(2) Good properties (i.e. smoothness) tend to be open in families (and often that open set is non-empty so most members of the family have this good property).

The next two lemmas and one theorem are proven in Shaf. I (page 141):

Assume that k has characteristic 0, X is smooth, and $X \overset{\pi}{\twoheadrightarrow} B$ is a family of varieties.

Lemma 2.4.3. *The fiber $\pi^{-1}(y)$ is nonsingular if $T_xX \xrightarrow{d_x\pi} T_yY$ is surjective $\forall x \in \pi^{-1}(y)$.*

Lemma 2.4.4. *There exists a nonempty open subset $V \subset X$ such that $d_x\pi$ is surjective $\forall x \in V$.*

Theorem 2.4.5. *Assume that k has characteristic 0. If $X \overset{\pi}{\twoheadrightarrow} B$ is a family of varieties where X is smooth, then the set $\{b \in B \mid \pi^{-1}(b)$ is smooth $\}$ is an open dense set of B.*

Example 2.4.6. if $V = \mathbb{V}(xy - z) \subseteq \mathbb{A}^3, SingV = \mathbb{V}(xy - z) \cap \mathbb{V}(y, x, -1) = \emptyset$, so V is smooth. $V \overset{\pi}{\twoheadrightarrow} \mathbb{A}^1$, A^1 is irreducible so by the theorem, $\{a \in A^1 \mid \pi^{-1}(b)$ is smooth $\}$.

Example 2.4.7. Fix $X \subseteq \mathbb{P}^n = \mathbb{P}(V)$ irreducible, projective variety. $\{X \cap H\}_{H \in \mathbb{P}(V^*)}$ is a family in the loose sense and in the technical sense. This is called the hyperplane section family. Locus of smooth hyperplane sections is open in $\mathbb{P}(V^*)$. So if one member is smooth, this locus is non-empty and hence most members are smooth. Why is this a family in the technical sense?
Let $\mathbb{X} = \{(p, H) \mid p \in H \cap X\} \subseteq \mathbb{P}(V) \times \mathbb{P}(V^*)$. Then $\mathbb{X} \twoheadrightarrow \mathbb{P}(V^*)$.

Theorem 2.4.8. *(Bertini's) If $X \subseteq \mathbb{P}^n$ is smooth then the general hyperplane section of X is smooth.*

The most useful kinds of families are (flat) the ones where the members "vary continuously."

Definition 2.4.9. A family $\mathbb{X} \overset{\pi}{\twoheadrightarrow} B$ is **flat** if there is an affine cover $\{U_i\}$ of B and an affine cover $\{V_{i,j}\}$ of each $\pi^{-1}(U_i)$ such that the induced map of affine varieties $V_{i,j} \overset{\pi|_{V_{i,j}}}{\twoheadrightarrow} U_i$ induces a flat map of algebras $\mathcal{O}_B(U_i) \longrightarrow \mathcal{O}_{\mathbb{X}}(V_{i,j})$.

Definition 2.4.10. $A \xrightarrow{f} B$ is a **flat map of rings** if for all short exact sequences of A-modules, $0 \to M_1 \to M_2 \to M_3 \to 0$, the induced sequence $0 \to B \otimes_A M_1 \to B \otimes_A M_2 \to B \otimes_A M_3 \to 0$, is exact.

Theorem 2.4.11. *(Hironaka Fields Medal) Every algebraic variety (over a field of characteristic 0) can be desingularized. More precisely, if V is a variety over a field k of characteristic 0, then there exists a smooth variety X and a projective birational morphism $X \twoheadrightarrow V$.*

Remarks on Hironaka's Theorem:

(1) The original proof is very hard - it took up two volumes of the Annals of Math. Now there is a "simple" proof that would only take about 6 weeks to teach in this class.
(2) The question is still open in characteristic $p > 0$.
(3) Hironaka's Theorem actually gives the resolution as a composition of easily understandable steps "blowing up along smooth subvarieties."

Definition 2.4.12. A **projective morphism** $X \longrightarrow V$ is one that factors as $X \overset{\text{closed}}{\hookrightarrow} V \times \mathbb{P}^n \xrightarrow{\pi_1} V$

Recall that a birational morphism is one that is an isomorphism on a dense open set.
In fact, $X \setminus \pi^{-1}(SingV) \underset{\cong}{\xrightarrow{\pi}} V \setminus SingV$.

Example 2.4.13. Let $V = \mathbb{V}(x^2+y^2-1) \subseteq \mathbb{A}^3$ and $W = \mathbb{V}(x^2+y^2-z^2) \subseteq \mathbb{A}^3$. Let $V \xrightarrow{f} W$ by sending $(x,y,z) \mapsto (xz, yz, z)$. This map sends a cylinder to a "cylinder" pinched in the middle (i.e. one singular point). So V is the desingularized version of W.

Lecture 24. A Beginner's Guide to Blow-ups

Remark 2.4.14. A smooth variety is similar to a manifold. In fact, if X is smooth over $\mathbb{C}$, then it is a complex manifold of the same dimension and a real (smooth) manifold of twice the dimension.

Theorem 2.4.15. *Let $X \xrightarrow{f} Y$ be a morphism of smooth varieties over a field of characteristic 0.*
The set $\{y \in Y \mid f^{-1}(y)$ is not a smooth variety $\}$ is a closed, proper subset of Y.

Remark 2.4.16. This theorem is the analogue to Sard's Theorem in differential topology.

Theorem 2.4.17. *(Sard's) If $X \xrightarrow{f} Y$ is a smooth map of smooth manifolds, then the set*
$\{y \in Y \mid f^{-1}(y)$ is not a manifold $\}$ has measure zero.

Theorem 2.4.18. *(Hironaka) If V is a variety over a field of characteristic 0, then there is a smooth variety $X \subseteq V \times \mathbb{P}^n$ closed and the projection $X \xrightarrow{\pi_1} V$ is an isomorphism $X \setminus \pi^{-1}(SingV) \xrightarrow[\cong]{\pi} V \setminus SingV$.*

Remark 2.4.19. Hironaka constructs X by a sequence of nice projections called blowing up.

Blowing up a point $p \in \mathbb{A}^2$
Let $p = (0,0)$. Let the coordinates of $\mathbb{A}^2$ be x, y and of $\mathbb{P}^1$ be $s : t$.
Viewing $\mathbb{P}^1$ as the set of lines in $\mathbb{A}^2$ through p define:

$$\begin{aligned} B_p\mathbb{A}^2 &= \{(x,l) \mid x \in l\} \subseteq \mathbb{A}^2 \times \mathbb{P}^1 \\ &= \{\big((x,y),[s:t]\big) \mid (x,y) = \lambda(s,t) \text{ for some } \lambda \in k\} \\ &= \mathbb{V}(xt-ys) \subseteq \mathbb{A}^2 \times \mathbb{P}^1 \end{aligned}$$

Define the projection $B_p\mathbb{A}^2 \xrightarrow{\pi} \mathbb{A}^2$ by sending $((x,y),l) \mapsto (x,y)$. The fiber over any non-zero point (x,y) is one point and the fiber over $(0,0)$ is $\pi^{-1}((0,0)) = \{((0,0),l) \mid (0,0) \in l\} = \{(0,0) \times \mathbb{P}^1\}$.
Look in an affine patch $\mathbb{A}^2 \times U_s = \mathbb{A}^2 \times \mathbb{A}^1 = \mathbb{A}^3$, where $U_s = \{\frac{t}{s} = z\}$, so that $B_p\mathbb{A}^2 = \mathbb{V}(xz-y)$.
Define the map from $B_p\mathbb{A}^2 \longrightarrow \mathbb{A}^2$ by $(x, y, \frac{y}{x}) \mapsto (x,y)$

Definition 2.4.20. The **blow-up** of $\mathbb{A}^2$ along $p \in \mathbb{A}^2$ is the projection morphism $B_p \xrightarrow{\pi} A^2$, where $\mathbb{P}^1$ is the set of lines through p in $\mathbb{A}^2$.

Observations:

(1) $B_p\mathbb{A}^2$ is a smooth, quasi-projective variety
(2) The projection morphism $B_p\mathbb{A}^2 \longrightarrow \mathbb{A}^2$ is projective and birational. There is also a map $B_p\mathbb{A}^2 \longleftarrow \mathbb{A}^2 \setminus \{p\}$ given by $(x,y) \mapsto \{\big((x,y),[x:y]\big)\}$.

Example 2.4.21. The curve $V = \mathbb{V}(y^2 - x^2 - x^3) \subseteq \mathbb{A}^2$ has a singularity only at the point $(0,0,0)$ and is called the "alpha" curve because it looks like an α when graphed in $\mathbb{R}^2$. By avoiding the singularity, we get an isomorphism, $\pi^{-1}(V \setminus (0,0)) \xrightarrow{\cong} V \setminus \{(0,0)\}$.

Lecture 25. Blowing-Up Along Varieties

Blowing up a point $p \in \mathbb{A}^n$
Let $p = (0, \dots, 0) \in \mathbb{A}^n$. Let the coordinates of $\mathbb{A}^n$ be $x_1, \dots, x_n$ and of $\mathbb{P}^{n-1}$ be $y_1 : \dots : y_n$. Viewing $\mathbb{P}^{n-1}$ as the set of lines in $\mathbb{A}^n$ through p define:

$$\begin{aligned} B_p(\mathbb{A}^n) &= \{((x_1, \dots, x_n), l) \mid (x_1, \dots, x_n) \in l\} \\ &= \mathbb{V}\Big(2 \times 2 \text{ minors of } \begin{pmatrix} x_1 \dots x_n \\ y_1 \dots y_n \end{pmatrix}\Big) \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}. \end{aligned}$$

The projection morphism $B_p\mathbb{A}^n \longrightarrow \mathbb{A}^n$ is projective and birational. There is also a map $B_p\mathbb{A}^n \longleftarrow \mathbb{A}^n \setminus \{p\}$ given by $(x_1, \dots, x_n) \mapsto \{\big((x_1, \dots, x_n), [x_1 : \dots : x_n]\big)\}$.

Easy Facts:
Let $V \xrightarrow{\varphi} Y$ be a regular map. Then the graph of φ: $\Gamma_\varphi = \{(x, \varphi(x))\} \subseteq V \times W$ is closed and $\Gamma_\varphi \underset{\pi}{\overset{\pi^{-1}}{\rightleftarrows}} V$ is an isomorphism.
These maps are mutually inverse regular maps by $(x, \varphi(x)) \leftrightarrow x$.

Now consider a rational map $V \overset{\varphi}{\dashrightarrow} W$ where V and W are irreducible.

Definition 2.4.22. The **graph of the rational map** $V \overset{\varphi}{\dashrightarrow} W$ is the closure in $V \times W$ of the graph of the regular map $U \xrightarrow{\varphi|_U} W$, where $U \subseteq V$ is open and dense. In particular, $\Gamma_\varphi = \overline{\{(X, \varphi(x)) \mid x \in U\}} \subseteq V \times W$.

Proposition 2.4.23. *For any rational map $V \overset{\varphi}{\dashrightarrow} W$ between irreducible varieties, $\Gamma_\varphi \xrightarrow{\pi} V$ is a birational morphism (i.e. it is an isomorphism over the domain of definition of φ).*

Remark 2.4.24. We have no control over π^{-1}(the locus of indeterminacy), it is some closed set in $V \times W$.

Proof. $V \times W \supseteq \Gamma_\varphi \xrightarrow{\pi} V$. For $U \subseteq V$, $\Gamma|_{\varphi|_U} = \Gamma_\varphi \cap (U \times W) \underset{\cong}{\xrightarrow{\pi}} U$ □

Think about blowing up a point in $\mathbb{A}^2$:
The map $\mathbb{A}^2 \overset{\varphi}{\dashrightarrow} \mathbb{P}^1$ sending $(x, y) \mapsto [x : y]$ is rational on $\mathbb{A}^2$ and regular on $\mathbb{A}^2 \setminus \{(0, 0)\}$.
$\Gamma_\varphi = \overline{\big(\{(x, y), [x : y]\big) \mid (x, y) \in \mathbb{A}^2 \setminus (0, 0)\}} = \mathbb{V}(xt - ys) \subseteq \mathbb{A}^2 \times \mathbb{P}^1$.
Note: If we let $p = (\lambda_1, \lambda_2)$, then we can look at $(x, y) \mapsto [x - \lambda_1 : y - \lambda_2]$ to see the blow-up at a point other than the origin. In this case we have $\Gamma_\varphi = \mathbb{V}((x - \lambda_1)t - (y - \lambda_2)s)$.

Summary: The blow-up $B_{\overline{0}}\mathbb{A}^n \longrightarrow \mathbb{A}^n$ can be interpreted as the graph of the rational map $\mathbb{A}^n \overset{\varphi}{\dashrightarrow} \mathbb{P}^{n-1}$, where $(x_1, \dots, x_n) \mapsto [x_1 : \dots : x_n]$, together with projection onto the first coordinate: $B_{\overline{0}}\mathbb{A}^n = \Gamma_\varphi \xrightarrow{\pi} \mathbb{A}^n$.

Definition 2.4.25. Let V be an affine variety, $W \subseteq V$ a closed subvariety. Say $\mathbb{I}(W) \subseteq k[V]$ has generators $F_0, \dots, F_t$. The **blow-up of** V **along** W, $B_W V$, is the graph of the rational map $V \dashrightarrow \mathbb{P}^t$ sending $\overline{x} \mapsto [F_0(\overline{x}) : \dots : F_t(\overline{x})]$, together with projection onto the first coordinate.

Remark 2.4.26. Up to isomorphism, this does not depend on our choice of generators $F_0, \dots, F_t$.

Now we have $V \times \mathbb{P}^t \supseteq B_W V \underset{\pi^{-1}}{\overset{\pi}{\rightleftarrows}} V$, where π is projection and $V \setminus W \xrightarrow{\pi^{-1}} B_W V$.

In general, how $B_W V$ looks is completely mysterious (except that it contains an open set isomorphic to $V \setminus W$). But when W is smooth, we do understand how $B_W V$ looks because then $V \times \mathbb{P}^t \supseteq B_W V \longrightarrow V$.

Definition 2.4.27. The **blow-up of an affine variety, V, along an ideal** $I = (F_0, \ldots, F_t)$, denoted by $B_I V$, is the graph of the rational map $V \dashrightarrow \mathbb{P}^t$ sending $x \mapsto [F_0(x) : \ldots : F_t(x)]$ together with projection onto V.

Remark 2.4.28. This is a projective, birational, isomorphism over $V \setminus \mathbb{V}(F_0, \ldots, F_t)$.

Another statement of Hironaka's Theorem:
Let V be an affine variety over a field k of characteristic 0. Let $B_I V$ be the graph of the maps $V \dashrightarrow \mathbb{P}^t$ sending $x \mapsto [F_0(x) : \cdots : F_t(x)]$. Then $\exists I = (F_0, \ldots, F_t) \subseteq k[V]$ such that $B_I V \longrightarrow V$ is a resolution of singularities of V. So $B_I V$ is smooth!

And yet another version:
Let V be a variety over a field k of characteristic 0. Then there exists a sequence of blowings up along smooth subvarieties which eventually terminates in a resolution of singular varieties of V. (We are presumably blowing up along a subvariety of $SingV$ at each step.

Lecture 26. Local Parameters

Lemma 2.4.29. *(Nakayama's) Let M be a finitely generated module over a local (Noetherian) ring R (with max ideal m). Then $m_1, \ldots, m_t \in M$ generate $M \Leftrightarrow$ their images, $\overline{m_1}, \ldots, \overline{m_t}$ span the R/m-vector space M/mM. In particular, they are a minimal generating set $\Leftrightarrow \overline{m_1}, \ldots, \overline{m_t}$ are a basis.*

Developing the idea of "local coordinates" at a smooth point $p \in X$ variety. Since X already has local coordinates inherited from affine space, we will call these "local coordinates", "parameters at p" instead.
Note that smooth, non-singular and simple are all equivalent.

Example 2.4.30. $p = (\lambda_1, \ldots, \lambda_n) \in \mathbb{A}^n$: local parameters at p are $u_1 = x_1 - \lambda_1, \ldots, u_n = x_n - \lambda_n$

Note:

(1) The $u_1, \ldots, u_n$ are n regular functions at p and $dim_p X = n$
(2) The u_i all vanish at p and they cut out precisely p (in a neighborhood of p)
(3) The max ideal in the local ring, $\mathcal{O}_{X,p}$ is generated by (the images of) $(u_1, \ldots, u_n)$
(4) The images, $\overline{u_1}, \ldots, \overline{u_n}$, in m/m^2 are a basis for the cotangent space since p is a smooth point implies that the dimension of the cotangent space is n. We can think of m/m^2 as a k-vector space (where $k \subset R = \mathcal{O}_{X,p}$), a R/m-vector space, or an R-module.

Definition 2.4.31. Let p be a smooth point on a variety X of dimension n (at p). Let $u_1, \ldots, u_n$ be regular functions at p that vanish at p. (Note: $u_i \in \mathcal{O}_{X,p}$). Then $u_1, \ldots, u_n$ are **parameters at** p if their images in m/m^2 are a basis for this cotangent space, where m is the maximal ideal of $\mathcal{O}_{X,p}$.

Equivalently, $u_1, \ldots, u_n$ are a minimal generating set for the $m \in \mathcal{O}_{X,p}$.

Remark 2.4.32. $\mathcal{O}_{X,p}$ is the local ring of X at p, so it is the ring of functions on X that are regular at p.

Example 2.4.33. Let $X = \mathbb{V}(x^2+y^2-1) \subseteq \mathbb{A}^2$ and $p = (0, 1)$. So $k[X] = k[x, y]/(x^2+y^2-1) \supseteq m_p = (x, y-1)$. Note that we get m_p from the local parameters at p, which are $x, y-1$.

Then

$$\begin{aligned}\mathcal{O}_{X,p} &= (k[x,y]/(x^2+y^2-1))(x,y-1)\\ &= \frac{k[x,y]}{x^2+y^2-1}[\big(\mathcal{O}_{X,p}\setminus(x,y-1)\big)^{-1}]\end{aligned}$$

Here we are inverting the non-zero set.
Claim: The max ideal of $\mathcal{O}_{X,p}$ is generated by x:
$(y-1)(y+1) = y^2-1 = -x^2$ in $k[X]_{m_p} \Rightarrow y-1 = x(\frac{-x}{y+1}) \in (x) \in \mathcal{O}_{X,p}$
So $m = (x, y-1) = (x) \Rightarrow x$ is a parameter at p.
Counting multiplicities, any line intersects X at two points, so p has multiplicity 2.

Remark 2.4.34. $R[U^{-1}] \equiv \{\frac{r}{v} \mid r \in R, v \in U\}$. Let $U = \{1, f, f^2, \ldots\} = (f)$. Then $R[U^{-1}] = R[\frac{1}{f}]$.
Let $P \subset R$ prime ideal and $U = R - P$. Then the standard notation for $R[U^{-1}]$ is R_p.

Theorem 2.4.35. *Local statement: Let $u_1, \ldots, u_n$ be parameters at a smooth point $p \in X$. Then the subvariety of X cut out by any subset of the u_i's is smooth at p. More precisely: There exists an open neighborhood $U \subseteq X$ of p where $u_1, \ldots, u_n$ are all regular and $p \in V \equiv \mathbb{V}(u_{i_1}, \ldots, u_{i_c}) \cap U \subseteq U \subseteq X$ so that V smooth at p.*

Proof. Without loss of generality, replace X by an open set. Let $V = \mathbb{V}(u_1, \ldots, u_c)$ for some $1 \le c \le n$ and take any point $p \in V$ that is a smooth point of X. Then $\mathcal{O}_{V,p} \twoheadleftarrow \mathcal{O}_{X,p}$ by restriction. Since $\mathcal{O}_{X,p}$ is generated by $u_1, \ldots, u_n$, $\mathcal{O}_{V,p}$ is generated by $u_1|_V, \ldots, u_n|_V \Rightarrow \mathcal{O}_{V,p}$ is generated by $\overline{u_c}, \ldots, \overline{u_n}$. Therefore:

$$n - c \le dim_p V \le dim(m_{V,p}/(m_{V,p})^2) = dim(\text{cotangent space to V at p}) \le n - c$$

Hence $dim_p V = n - c$ and p is smooth of V. □

3. Rational Maps, Regular Maps, and Divisors Thereof

Lecture 27. Local Defining Equations

Goal: Study rational and regular maps from a variety to a projective space.

Set-up: Let X be an irreducible variety and $X \overset{\varphi}{\dashrightarrow} \mathbb{P}^n$ a rational map sending $x \mapsto [\varphi_0(x) : \cdots : \varphi_n(x)]$, where $\varphi_i \in k(X)$. Each $\varphi = \frac{f_i}{g_i}$ where f_i, g_i are regular at p. By clearing denominators, we can assume that all of the φ_i are regular at $p \in X$.

Example 3.0.36. The map $\mathbb{A}^2 \dashrightarrow \mathbb{P}^1$ that sends $(x, y) \mapsto [x : y]$ is rational with domain of definition $\mathbb{A}^2 - \{(0,0)\}$ and locus of indeterminacy the point $(0,0)$.

Example 3.0.37. Let $V = \mathbb{V}(x^2 + y^2 - z^2) \subset \mathbb{P}^2$ and $V \overset{\varphi}{\dashrightarrow} \mathbb{P}^1$ by sending $[x : y : z] \mapsto [x : y - z](= [\frac{x}{z} : \frac{y}{z} - 1]$ on $U_z)$. For $[x : y : z] \in V$, $[x : y - z] = [x(y + z) : (y - z)(y + z)] = [x(y + z) : -x^2] = [y + z : -x]$. We obtain φ by stereographically projecting from $[0 : 1 : 1] = \mathbb{V}(x, y - z)$. When we write φ as $\varphi([x : y : z]) = [\frac{x}{z} : \frac{y}{z} - 1]$, φ is regular on $X \setminus [0 : 1 : 1]$.

Theorem 3.0.38. ***Thm 1*** *Let X be a smooth irreducible variety and $X \overset{\varphi}{\dashrightarrow} \mathbb{P}^n$ be a rational map sending $x \mapsto [\varphi_0(x) : \cdots : \varphi_n(x)]$. Then the locus of indeterminacy has codimension at least 2.*

Remark 3.0.39. For instance, if X has dimension 2, then there are at most finitely many points where φ is not regular. This is because the locus of indeterminacy has dimension 0 by the theorem.

Corollary 3.0.40. *If X is a smooth curve, every rational map $X \overset{\varphi}{\dashrightarrow} \mathbb{P}^n$ is regular at all points.*

Corollary 3.0.41. *For smooth projective curves, birational equivalence is the same as isomorphism.*

Remark 3.0.42. This is false in higher dimensions. For instance, $\mathbb{P}^2 \sim \mathbb{P}^1 \times \mathbb{P}^1 \sim B_p\mathbb{P}^2 \sim \ldots$ etc.

Definition 3.0.43. $f_1, \ldots, f_t \in \mathcal{O}_{X,p}$ are **local defining equations** for closed $Y \subsetneq X$ containing the point p if $\exists U \subseteq X$ affine open neighborhood of p where $f_1, \ldots, f_t$ are regular and $\mathbb{I}(Y \cap U) \subseteq \mathcal{O}_X(U)$ is generated by the $f_1, \ldots, f_t$. Equivalently, if $\{g \in \mathcal{O}_{X,p} \mid g$ vanishes in Y on an open neighborhood of $p\} = \mathbb{I}(V)_p = \mathcal{O}_{X,p}$ is generated by $f_1, \ldots, f_t$.

Example 3.0.44. Let $f = x^2 + y^2 - z^2$ and $Y = \mathbb{V}(f) \subseteq \mathbb{P}^2$. At $p = [0 : 1 : 1], f$ is a local defining equation for $Y \cap U_z$ at every point of U_z (refer to example of this earlier to see this).

Recall: If $Y \subseteq \mathbb{A}^n$ has (pure) codimension 1, then $\mathbb{I}(Y) = (f)$ for some $f \in k[x_1, \ldots, x_n]$. So f is a local defining equation for Y at all points of $\mathbb{A}^n$.
Question: If $Y \subseteq X$ is closed with pure codimension 1, does Y always have one local defining equation? **No.**

Example 3.0.45. Let $Y = \mathbb{V}(x, y)$ and $X = \mathbb{V}(xz - y^2)$ so that $Y \subseteq X \subseteq \mathbb{A}^3$. Then $(x, y) = \mathbb{I}(Y) \subseteq k[X] = \frac{k[x,y,z]}{(xz-y^2)}$, which implies that $x = \frac{y^2}{z}$ and so $(x, y) = (y)$ in $\mathcal{O}_X(U_z)$. Hence y is a local defining equation in U_z (and U_x by symmetry). But at $p = (0,0,0)$, we need both x and y to generate $\mathbb{I}(Y)$ in $\mathcal{O}_{X,p}$.
$\mathbb{I}(Y)$ *cannot be generated by one element:* Take any $u, v \in \frac{k[x,y,z]}{(xz-y^2)}$ where v has a non-zero

constant term, such that $(x,y)=(\frac{u}{v})$ in $\frac{k[x,y,z]}{(xz-y^2)}(x,y,z)=\mathcal{O}_{X,p}$ (that is, invert everything that is outside of (x,y,z)).
Lifting to $k[x,y,z]$ we get that $xv-u\in(xz-y^2)$ and $yv-u\in(xz-y^2)$.
Restricting to the terms of degree 1: $xv_0-u_1\in(xz-y^2)\Rightarrow xv_0=u_1$ and, likewise, $yv_0=u_1$. Therefore $xv_0=yv_0=u_1$. This is a contradiction since $x\neq y$.

Theorem 3.0.46. ***Thm 2:*** *If $Y\subseteq X$ is a closed subset of X of codimension 1 and $p\in X$ is a smooth point, then Y has a local defining equation at p.*

In particular, there exists an open affine set $U\subset X$ containing p such that $\mathbb{I}(Y)$ is principal on U. More precisely, if $Y\cap U\subseteq U$ is a closed subset of an affine variety, then $\mathbb{I}(Y\cap U)\subseteq\mathcal{O}_X(U)$ is principal or, equivalently, $\mathbb{I}(Y)$ is principal near any smooth point $p\in X$.

The main algebraic fact underlying **Thm 1** is:

Theorem 3.0.47. ***Thm 3*** *The local ring $\mathcal{O}_{X,p}$ of a smooth point p on a variety X is a UFD.*

The idea behind the proof **thm 3** is: $\mathcal{O}_{X,p}\hookrightarrow\hat{\sigma}_{X,p}=\{$ completion of $\mathcal{O}_{X,p}$ at its maximal ideal $\}=k[[u_1,\dots,u_s]]$, where u_i are parameters at p. So any regular function at p can be written as a power series.

Lecture 28.

Example 3.0.48. Let $\mathbb{A}^n\overset{\varphi}{\dashrightarrow}\mathbb{P}^t$ be given by $x\mapsto[F_0(x):\cdots:F_t(x)]$. We may assume that the F_i are polynomials with no common factors by clearing denominators and using the property that $k[x_1,\dots,x_n]$ is a UFD to discard all common factors. Since φ is defined on (at least) $\mathbb{A}^n-\mathbb{V}(F_0,\dots,F_n)$,

$$\big(\text{locus of indeterminacy of }\varphi\big)\subseteq\mathbb{V}(F_0,\dots,F_t)\text{ and}$$
$$codim\big(\text{locus of indeterminacy of }\varphi\big)\geq codim(\mathbb{V}(F_0,\dots,F_t))$$

Claim: $codim\mathbb{V}(F_0,\dots,F_t)\geq 2$.
Suppose not. Then there is an irreducible variety Y of codimension 1 with $Y\subseteq\mathbb{V}(F_0,\dots,F_t)$ and $\mathbb{I}(Y)=(g)$ for some irreducible polynomial. Therefore g divides all the F_i, which is a contradiction.
Note: Suppose $\mathbb{I}(Y)=(g_1,\dots,g_s)$. $\mathbb{I}(Y)$ is a prime ideal and so some irreducible factor g of g_1 is in $\mathbb{I}(Y)$. Then $(g)\subseteq\mathbb{I}(Y)$, $\mathbb{A}^n\supsetneq\mathbb{V}(g)\supseteq Y$, and Y has codimension 1. Therefore $\mathbb{I}(Y)=\mathbb{I}(\mathbb{V}(g))=\mathbb{I}(g)$.

Thm 1 ⇒ Thm 3:

Proof. Use the fact that height one ideals in a UFD are principal. □

Thm 3 ⇒ Thm 2:

Proof. Without loss of generality, we can assume that X is affine. Take $Y\subseteq X$ and $p\in X$ satisfying the conditions for Thm 2. Look at $\mathbb{I}(Y)=(g_1,\dots,g_s)\subseteq k[X]$, which has codimension c in $\mathcal{O}_{X,p}$. Then $\mathbb{I}(Y)_p\subseteq\mathcal{O}_{X,p}$ is a UFD by theorem 3. We have 2 cases:
Case 1 Y is irreducible:
Then $(g_1,\dots,g_s)$ factors into irreducibles and is prime $\Rightarrow$ we may assume that the $g_i's$ are all irreducible. $(g_1)\subseteq(g_1,\dots,g_s)\Rightarrow\mathbb{V}(g_1)\subseteq\mathbb{V}(g_1,\dots,g_s)$ and the latter has codimension one $\Rightarrow\mathbb{V}(g_1)=\mathbb{V}(g_1,\dots,g_s)$.
Case 2 $Y=Y_1\cup\ldots\cup Y_r$, where each Y_i is irreducible:

By case 1, $\mathbb{I}(Y_i) = (g_i)$ near $p \Rightarrow \mathbb{I}(Y) = (g_1, \dots, g_r)$ near p. This is radical because the g_i are all distinct and irreducible in a UFD. □

Thm 1:

Proof. $X \overset{\varphi}{\dashrightarrow} \mathbb{P}^t$ where $x \mapsto [\varphi_0(x) : \dots : \varphi_t(x)]$ and $\varphi_i \in k(X)$. Let Y be the locus of indeterminacy of φ. Suppose, by way of contradiction, that Y has a component of codimension one.
Pick p on such a component and consider φ near p. Clear denominators so that we map assume that $\varphi_i \in \mathcal{O}_{X,p}$ Cancel to get that the φ_i's have no common factors in $\mathcal{O}_{X,p}$. Our codimension 1 component defined by one equation ψ near p. φ defined outside of $\mathbb{V}(\varphi_0, \dots, \varphi_t) \Rightarrow$ our component is a subset of $\mathbb{V}(\varphi_0, \dots, \varphi_t) \Rightarrow \mathbb{I}($ our component$)=$ $(\psi) \supseteq (\varphi_0, \dots, \varphi_t) \Rightarrow \psi$ divides each of the $\varphi_0, \dots, \varphi_t$.
This is a contradiction. Note that this proof is very similar to the proof of our earlier example. □

Convenient Notation:
For a variety X, we have the structure sheaf, $\mathcal{O}_X$
$\mathcal{O}_X \rightsquigarrow U \subseteq X, \mathcal{O}_X(U) =$ rational functions regular on U
If we have $Y \subseteq X$ a closed subvariety, then:
$\mathbb{I}(Y) \rightsquigarrow \mathbb{I}(Y)(U) =$ the set of functions in $\mathcal{O}_X(U)$ that vanish on $Y \cap U$
$\mathcal{O}_X$ is a sheaf and $\mathbb{I}(Y)$ is the sheaf of ideals of Y.

Remark 3.0.49. What theorem 2 really says is that the sheaf of ideals of a codimension one subvariety on a smooth variety is locally principal.

3.1. **Divisors.**

Lecture 29. Introducing Divisors

Algebra Blackbox:
Corollary 3.1.1. *If R is a Noetherian doman, $I \subsetneq R$ ideal, then* $\cap_t I^t = 0$

For today, fix X irreducible variety over k

Definition 3.1.2. A **prime divisor** of X is an irreducible codimension 1 (closed) subvariety.

Definition 3.1.3. A **divisor** D on X is a finite formal $\mathbb{Z}$-linear combination of prime divisors: $D = \Sigma_i n_i Y_i$, where $n_i \in \mathbb{Z}$ and Y_i is irreducible of codimension 1 in X.

Definition 3.1.4. $Div(X)$ is the free abelian group generated by prime divisors on X

Example 3.1.5. Let $X = \mathbb{A}^1$, a point $[\lambda]$ where $\lambda \in k$ is a prime divisor. A divisor on $\mathbb{A}^1$ is $\Sigma_{i=1}^t n_i[\lambda_i]$.

Example 3.1.6. Let $X = \mathbb{P}^2$, a prime divisor is an irreducible curve $\mathbb{V}(f)$, where f is irreducible and homogeneous in x, y, z. For example, $L = \mathbb{V}(\text{linear})$ and $C = \mathbb{V}(x^2 + y^2 - z^2)$. A divisor would be a finite sum of irreducible curves. For example, $2L - C$.

Example 3.1.7. Let $X \subseteq \mathbb{P}^n$ be irreducible and not contained in any hyperplane. Then $X \cap H$ has codimension 1 for any hyperplane H. For example, let $X = \mathbb{V}(x^2 + y^2 - z^2) \subseteq \mathbb{P}^2$ - note that on this curve, some points have multiplicity 2.

Example 3.1.8. Let $f = \frac{g}{h} \in k(\mathbb{A}^n)$, where $g, h \in k[x_1, \dots, x_n]$. There is a divisor "of zero's and poles" of f. If $\frac{g}{h} = \frac{g_1^{a_1} \dots g_r^{a_r}}{h_1^{b_1} \dots h_s^{b_s}}$ where $g_i, h_j \in k[x_1, \dots, x_n]$ are irreducible, then:
$div(f) = div(\frac{g}{h}) =$ the divisor of zeros of $\Sigma a_i \mathbb{V}(g_i) -$ the divisors of poles $\Sigma b_i \mathbb{V}(h_i)$.

Example 3.1.9. Let $X \xrightarrow{\varphi} \mathbb{P}^1$ be a regular map. Then φ has fibers of codimension 1, so $\mathbb{V}(\varphi)$ is a divisor on X. In particular, any regular map is a divisor in this scenario.

Remark 3.1.10. Regular and rational maps to projective space are governed by families of divisors.

Definition 3.1.11. Let $U \subset X$ be an open affine set and let $g \in \mathcal{O}_X(U)$. Let $Y \subseteq \mathbb{V}(g)$ be irreducible. The **order of vanishing** of g along Y is $t \in \mathbb{N}$ such that $g \in (\pi_i)^t - (\pi_i)^{t+1}$. Notation: $t = \nu_{Y_i}(g) = ord_{Y_i}(g)$.

Goal: Given any non-zero rational function $f \in k(X)$, we want $div(f)$ to be the "domain of zeros and poles." First assume that X is smooth and take an open affine set $U \subseteq X$. Then $f = \frac{g}{h}$, where $g, h \in \mathcal{O}_X(U')$. for some open affine set $U' \subseteq U$. Note that we need to choose a large enough open set U such that it is not the complement of a divisor. For instance, we can assume that $codim(X - U) \geq 2$.

$$\mathbb{V}(g) \text{ is a codimension 1 subvariety that equals } Y_1 \cup \cdots \cup Y_t, \text{ where the } Y_i \text{ are irreducible}$$
$$\mathbb{V}(h) \text{ is a codimension 1 subvariety that equals } Z_1 \cup \cdots \cup Z_s, \text{ where the } Z_i \text{ are irreducible}$$

We know that $div(f) = \Sigma a_i Y_i - \Sigma b_j Z_j$, but how do we find a_i, b_j?
Consider g, Shrink U if necessary so that Y_i is principal on U. Then $g \in \mathbb{I}(Y_i \cap U) = (\pi_i) \subseteq \mathcal{O}_X(U)$.
Define $div(f) = \Sigma a_i \overline{Y_i} - \Sigma b_i \overline{Z_i}$, where $a_i = ord_{Y_i}(g)$ and $b_i = ord_{Z_i}(h)$.
We need to check that this definition is:

1. Independent of our original choice of U and the way we wrote f as $\frac{g}{h}$
2. Independent of the smaller neighborhood, U', that we took to make the components locally principal
3. Independent of our choice of generator π

Example 3.1.12. Rational functions $f = \frac{F}{G}$, where F, G are homogeneous degree d polynomials in $\mathbb{P}^2$.
Let $F = F_1^{a_1} \dots F_r^{a_r}$ and $G = G_1^{b_1} \dots G_s^{b_s}$, where the F_i, G_i are irreducible and homogeneous. Then, $div(f) = \Sigma a_i \mathbb{V}(F_i) - \Sigma b_i \mathbb{V}(G_i)$.

Example 3.1.13. Let $f = \frac{x^2}{x^2+y^2-z^2}$. On the affine patch U_z we can replace x by $\frac{x}{z}, y$ by $\frac{y}{z}$, and z by 1 so that $f = \frac{(\frac{x}{z})^2}{(\frac{x}{z})^2+(\frac{y}{z})^2-1}$ where the numerator is F and the denominator is G, each of which is regular in $\mathcal{O}_{\mathbb{P}^2}(U_z)$. Then: $div(f) = a\mathbb{V}((\frac{x}{z})^2) - b\mathbb{V}((\frac{x}{z})^2 + (\frac{y}{z})^2 - 1) = aL - bC$ for some $a, b \in \mathbb{Z}$
$\mathbb{I}(L \cap U_z) = (\frac{x}{z})$ and $(\frac{x}{z})^2 \in (\frac{x}{z})^2 \setminus (\frac{x}{z})^3 \Rightarrow \nu_L((\frac{x}{z}) = 2 \Rightarrow a = 2$.
$\mathbb{I}(C \cap U_z) = (\frac{x}{z})^2 + (\frac{y}{z})^2 - 1) \Rightarrow \nu_C(\frac{x}{z})^2 + (\frac{y}{z})^2 - 1) = 1 \Rightarrow b = 1$.

$k(X)^* \xrightarrow{div} Div(X)$ by: $f \mapsto div(f)$ and $f_1 f_2 \mapsto div(f_1 f_2) = div(f_1) + div(f_2)$ is a group homomorphism.

Definition 3.1.14. The **group of principal divisors** of X, $P(X)$, is the image of the above map.

Definition 3.1.15. The **divisor class group** of (an irreducible variety) X, $Cl(X)$, is the group $Div(X)/P(X))$.

So $k(X)^* \longrightarrow Div(X) \twoheadrightarrow Cl(X)$ is an exact sequence.

Example 3.1.16. $Cl(\mathbb{A}^n) = 0$: The irreducible codimension 1 subvarieties of $\mathbb{A}^n$ are of the form $\mathbb{V}(f_i)$, where f_i is an irreducible polynomial. So any element $D \in Div(\mathbb{A}^n)$ is of the form:

$$D = \Sigma n_i \mathbb{V}(f_i) - \Sigma m_i \mathbb{V}(g_i) = div(\frac{f_1^{n_1} \dots f_t^{n_t}}{g_1^{m_1} \dots g_s^{m_s}}) \in P(X)$$

.

Example 3.1.17. $Cl(\mathbb{P}^n) \cong \mathbb{Z}$: Any $f \in k(\mathbb{P}^n)^*$ can be written as $f = \frac{F_1^{a_1} \dots F_r^{a_r}}{G_1^{b_1} \dots G_s^{b_s}}$, where $F_i, G_j \in k[x]$ and G_j non-zero. Then $div(f) = \Sigma a_i \mathbb{V}(F_i) - \Sigma b_j \mathbb{V}(G_j)$. Since $f \in k(\mathbb{P}^n)^*$, $\Sigma a_i deg F_i = \Sigma b_i deg G_i$. Then $Div(\mathbb{P}^n) \longrightarrow \mathbb{Z}$ is given by the degree so that $\Sigma n_i \mathbb{V}(H_i) \mapsto \Sigma n_i deg(H_i)$. The kernel of this map is $P(X)$. This gives us a short exact sequence:

$$0 \longrightarrow k(\mathbb{P}^n)^* \xrightarrow{div} Div(\mathbb{P}^n) \xrightarrow{degree} \mathbb{Z} \longrightarrow 0 \Rightarrow Cl(\mathbb{P}^n) \cong \mathbb{Z}$$

.

Remark 3.1.18. $Cl(X)$ is an invariant: $X \cong Y \Rightarrow Cl(Y) = Cl(X)$.

Lecture 30.

Algebra Blackbox:

Definition 3.1.19. A domain R is **normal** if it is integrally closed in its fraction field.

Remark 3.1.20. R is normal $\Leftrightarrow R_p$ is normal $\forall p \in SpecR \Leftrightarrow R_m$ normal $\forall m \in maxSpecR$.

Proposition 3.1.21. *The following are equivalent for any one dimensional local Noetherian domain (R, m):*

(1) *R is normal*
(2) *m/m^2 is a one dimensional R/m vector space*
(3) *m is principal, say $m = (\pi)$*
(4) *Every element of R can be written (unit)$(\pi)^t$ for some (uniquely determined) $t \in \mathbb{N}$. Note that $\nu(f) = t$ where $f =$(unit)π^t.*

Definition 3.1.22. Such a domain is called a **discrete valuation ring** (DVR).

For today, X is an irreducible variety over k.

Definition 3.1.23. Let $D = \Sigma_{i=1}^t n_i Y_i$ be a divisor where $n_i \neq 0$ and Y_i is a pure codimension one subvariety of X. The **support of** D, $SuppD$, is $Y_1 \cup \dots \cup Y_t$.

Definition 3.1.24. A divisor $D = \Sigma_i n_i Y_i$ of X is **effective** if all of the $n_i \geq 0$.

Let X be a variety with $SingX$ of codimension atleast two. Then every prime divisor Y must intersect the smooth locus, $X \setminus SingX$. Take an open affine $U \subseteq X \setminus SingX$ such that $\mathbb{I}(U \cap Y) = (\pi) \subseteq \mathcal{O}_X(U)$. Let $f \in \mathcal{O}_X(U)$. Then $\nu_Y(f) = t$ where $f \in (\pi^t) \setminus (\pi^{t+1})$.

Why is $\Sigma_Y \nu_Y(f) Y)$, where all of the Y are prime divisors, a finite sum?
Choose an open affine set U on which f is regular. Suppose $U \cap Y \neq \emptyset$, then:
$\nu_Y(f) \geq 0$ on U and $\nu_Y(f) > 0 \Leftrightarrow f$ vanishes along $Y \Leftrightarrow Y \subseteq \mathbb{V}(f) = Y_1 \cup \dots \cup Y_t$ on U where each of the Y_i has codimension 1 and so $\mathbb{V}(f)$ has codimension 1 $\Rightarrow Y$ is one of these finitely many Y_i.
Now suppose $U \cap Y = \emptyset \Rightarrow Y \subseteq X - U = Z_1 \cup \dots \cup Z_t \Rightarrow Y \subseteq Z_i \Rightarrow Y$ is one of the Z_i.
Since there are only finitely many irreducible components, the sum must be finite.

Notation: Let X be a variety and $Y \subseteq X$ irreducible. Then $I_Y(U) = \mathbb{I}(Y \cap U)$, where $U \subset X$ open affine set.

Definition 3.1.25. Let $Z \subseteq X$ be any irreducible closed subvariety of X. The **local ring of X along Z** is:

$$\begin{aligned}\mathcal{O}_{X,Z} &= \{\varphi \in k(X) \mid \varphi \text{ is regular at some point of } Z\}\\ &= \{\varphi \in k(X) \mid \text{the domain of definition of}(\varphi) \cap Z \neq \emptyset\}\\ &= \{\varphi \in k(X) \mid \varphi \text{ is regular on a non-empty open set of } Z\}\\ &= \varinjlim_{U\cap Z\neq\emptyset} \mathcal{O}_X(U)\\ &= \mathcal{O}_X(U)[f^{-1} \mid f \notin \mathbb{I}(Z\cap U)]\\ &= \mathcal{O}_X(U)_{\mathbb{I}(Z\cap U)}\end{aligned}$$

We are allowing rational functions whose denominator does not vanish completely along Z. In the last three equalities we passed to any open affine $U \subseteq X, U \cap Z \neq \emptyset$.

Example 3.1.26. If $Z = \{x\}, \mathcal{O}_{X,Z} = \mathcal{O}_{X,x} =$ local ring at x.

Suppose $Z = Y \subseteq X$ is codimension 1 and irreducible. Then $0 \subsetneq \mathbb{I}_Z \subseteq \mathcal{O}_X$, 0 is the only prime ideal contained in $\mathbb{I}_Z$. Note that it does not matter what open affine set we choose so we can write $\mathcal{O}_X$ instead of $\mathcal{O}_X(U)$. So $\mathbb{I}_Z$ has height 1. $\mathcal{O}_{X,\mathbb{I}_Z}$ has 2 prime ideals: (0) and $\mathbb{I}_Z\mathcal{O}_{X,Z}$. So $\mathcal{O}_{X,Z}$ is one dimensional.
If codimension of $SingX$ is atleast 2, then there is an open affine set U where $U \cap Z \neq 0$ and $\mathbb{I}_Z(U \cap Z)$ is principal.
If $SingX$ has codimension atleast 2 (if you prefer, you can think of X as being smooth), then $\mathcal{O}_{X,Y}$ is a discrete valuation ring (in $k(X)$) for all prime divisors, Y. Now, given $f \in k(X)^*, \nu_Y(f) =$ the valuation of f thinking of it as an element of the fraction field of $\mathcal{O}_{X,Y}$, which is a discrete valuation ring.

Definition 3.1.27. A variety X is **normal** if $\exists\{U_\lambda\}$ open affine cover of X such that $\mathcal{O}_X(U_\lambda)$ is normal.

Note that in this definition we are still assuming that X is irreducible. Equivalently we could define this as:

Definition 3.1.28. A variety X is **normal** if $\mathcal{O}_{X,x}$ is normal $\forall x \in X$.

This second definition makes it clear that normality does not depend on the affine cover we choose.

Lecture 31. Locally Principal Divisors

Algebra Blackbox: **Proposition 3.1.29.** *Every UFD is normal.* **Theorem 3.1.30.** *Let A be a normal domain, then $A = \cap_\Lambda A_p$, where $\Lambda = \{P \subset A$ prime ideal of height 1$\}$*

Corollary 3.1.31. *Every smooth variety is normal.*

Proof. X is smooth $\Rightarrow \mathcal{O}_{X,x}$ is a UFD $\forall x \in X \Rightarrow \mathcal{O}_{X,x}$ is normal $\Rightarrow X$ is normal. □

"Pop Quiz"
On $\mathbb{P}^n$, find four different divisors all representing the same class in $Cl(\mathbb{P}^n)$ but satisfying the four conditions:

(1) $SuppD$ has 1 irreducible component and its coefficient is 1:
$D = \mathbb{V}(F_d)$ where F_d is any irreducible degree d homogeneous polynomial

(2) $SuppD$ has d irreducible components:
$D = H_1 + \cdots + H_d$, *where the* H_i *are all distinct hyperplanes.*
(3) $SuppD$ is a hyperplane:
$D = H$, *where* H *is a hyperplane*
(4) D is not effective:
$\mathbb{V}(F_d) + H_1 - H_2$

Given $f \in k(X)^*$, how can we tell in terms of $div(f)$ whether or not f is regular?

Proposition 3.1.32. *Let X be normal, $f \in k(X)^*$. Then f is regular on $X \Leftrightarrow divf \geq 0$ (i.e. $divf$ is effective).*

Proof. ($\Rightarrow$) Suppose f is regular $\Rightarrow divf \geq 0$. Let Y be a prime divisor. To compute $\nu_Y(f)$, take any open affine set $U \subseteq X$ such that $U \cap Y \neq \emptyset, f \in \mathcal{O}_X(U) \rightsquigarrow f \in \mathcal{O}_{X,Y} \Rightarrow \nu_Y(f) \geq 0$.
($\Leftarrow$) Suppose $div(f) \geq 0 \Rightarrow \nu_Y(f) \geq 0 \forall Y$ prime divisors. To show f is regular, check $f|_U$ is regular $\forall U \subseteq X$ affine open. So $div(f) = \Sigma_Y \nu_Y(f)Y, f \in \mathcal{O}_{X,Y}(U) \Rightarrow f \in \cap_{Y \cap U \neq \emptyset} \mathcal{O}_{X,Y} = \mathcal{O}_X(U)$. Therefore f is regular on each $U \Rightarrow f$ is regular on X. □

What is the kernel of $k(X)^* \xrightarrow{div} Div(X)$?

Proposition 3.1.33. *$Ker(div) = \{f \in k(X)^* \mid f, f^{-1} \in \mathcal{O}_X(X)\} = \mathcal{O}_X^*(X) =$ the subgroup of invertible elements of $\mathcal{O}_X(X)$. This is because elements of $Ker(div)$ have no poles and no zeros.*

Example 3.1.34. If $X = \mathbb{A}^1 \setminus \{0\}$, then $\frac{1}{x} \in \mathcal{O}^*_{\mathbb{A}^1}(X)$.

Locally Principal Divisors: Let X be irreducible and normal.

Definition 3.1.35. Let $D = \Sigma_{i=1}^t n_i Y_i$ where the Y_i are prime. Then $D \in Div(X)$ is **locally principal** if the ideal $\mathbb{I}_{Y_i} \subseteq \mathcal{O}_X$ is locally principal. In particular, if there exists an open affine cover $\{U_\lambda\}$ of X such that $\mathbb{I}(Y_i \cap U_\lambda) \subseteq \mathcal{O}_X(U_\lambda)$ is principal.

Remark 3.1.36. If X is smooth, then every divisor is locally principal.

Example 3.1.37. Let $X = \mathbb{V}(xz - y^2) \subseteq \mathbb{A}^3$ and $L = \mathbb{V}(x, y) \subseteq X$. Then X is normal and L is a prime divisor that is not locally principal. For some $a \in \mathbb{Z}, div(X) = aL$. Since $\mathcal{O}_{X,L} = \frac{k[x,y,z]}{(xz-y^2)}(x, y)$, $\mathcal{O}_{X,L}$ has z as a unit and (y) as a maximal ideal. Now $x \in (y^2) \setminus (y^3)$ so $a = \nu_L(x) = 2$ and, therefore, $div(X) = 2L$.

Say that D is a locally principal divisor, $D = \Sigma_{i=1}^t n_i Y_i$. There is an open cover $\{U_\lambda\}$ such that $\mathbb{I}(Y_i \cap U_\lambda) = \mathbb{V}(\pi_{i\lambda})$. On $U_\lambda \rightsquigarrow D \cap U_\lambda = div(\pi_{1\lambda}^{n_1} \ldots \pi_{t\lambda}^{n_t} \equiv f_\lambda)$, where $f_\lambda \in k(X)^*$. We can think of a locally principal divisor D as data $\{(U_\lambda, f_\lambda)\}_\lambda$ where $f_\lambda \in k(X)^*$ and when $\{U_\lambda\}$ is an open cover of X. $div(f_\lambda \cap U_{\lambda'}) = (D \cap U_\lambda) \cap U_{\lambda'} = (D \cap U'_\lambda) \cap U_\lambda = div(f_{\lambda'} \cap U_\lambda)$.
So $div(f_\lambda f_{\lambda}^{-1}) = divf_\lambda - divf_{\lambda'} = 0$ on $U_\lambda \cap U_{\lambda'}$.
Therefore the f_λ satisfy $f_\lambda f_{\lambda'}^{-1} \in \mathcal{O}_X^*(U_\lambda \cap U_{\lambda'})$.

Lecture 32.

For this lecture, let X be a normal, irreducible variety.
$Div(X)$ is the free abelian group on irreducible codimension 1 subvarieties.
$Div(X) \supseteq CDiv(X)$ is the subgroup of locally principal divisors (i.e. Cartier diviors).
$CDiv(X) \supseteq P(X)$ is the subgroup of principal divisors $\{div(f)\}_{f \in k(X)^*}$.

Example 3.1.38. Let $X = \mathbb{P}^n$ and $D = \mathbb{V}(F_d)$, where F_d is an irreducible homogeneous degree $d > 0$ polynomial. $U_i = X - H_i$, where $H_i = \mathbb{V}(x_i)$. Then $D \cap U_i = div(\frac{F_d}{x_i^d}) = div(f_i)$, where f_i is a rational function on $\mathbb{P}^n$ that agrees with $\frac{F_d}{x_i^d}$. This is an example of a locally principal but not principal divisor.

Note that $CDiv(X)$ is a group since if we take $D_1 = \{U_i, divf_i\}$ and $D_2 = \{V_j, divg_j\}$ then $D_1 + D_2 = \{U_i \cap V_j, div(f_i g_j)\}_{i,j}$.

Definition 3.1.39. A **Cartier divisor** D on X is equivalent to the data $\{U_\lambda, f_\lambda\}$ where:

(1) $\{U_\lambda\}$ is an open cover of X
(2) $f_\lambda \in k(X)^*$
(3) $\frac{f_\lambda}{f_\eta} \in \mathcal{O}^*_X(U_\lambda \cap U_\eta)$, invertible regular functons on $U_\lambda \cap U_\eta$

Remark 3.1.40. This definition of Cartier divisor is the type that you find in textbooks because it makes sense "even if you are not normal." Whereas the typical definition that geometers use depends on being normal.

Two collections $\{U_\lambda, f_\lambda\}$ and $\{V_\mu, g_\mu\}$ define the same divisor $\Leftrightarrow f_\lambda \cdot g_\mu^{-1} \in \mathcal{O}^*_X(U_\lambda \cap V_\mu)$.

Definition 3.1.41. The **Picard group** of X is the quotient $\frac{CDiv(X)}{P(X)} \equiv Pic(X)$.

Note that $Pic(X) = \frac{CDiv(X)}{P(X)} \subseteq \frac{Div(X)}{P(X)} = Cl(X)$. if X is smooth, all divisors are locally principal, so $Div(X) = CDiv(X)$ and $Cl(X) = Pic(X)$. For example, $Pic(\mathbb{P}^n) = Cl(\mathbb{P}^n) = \mathbb{Z}$.

Let $X \xrightarrow{\varphi} Y$ be a regular map of irreducible varieties. Is there an induced map of class groups? Is there an induced map of Picard groups?
The Picard group defines a type of invariance between varieties.
Given a regular map $X \xrightarrow{\varphi} Y$, we want to define the pull-back of φ on divisors: $Div(Y) \xrightarrow{\varphi^*} Div(X)$ by sending $D = \Sigma n_i Y_i \mapsto \varphi^*(D) = \Sigma n_i \varphi^*(Y_i)$. But what is this map?

Example 3.1.42. Let $X = \mathbb{V}(x^2 + y^2 - 1) \subseteq \mathbb{A}^2$ where $X \xrightarrow{\pi} \mathbb{A}^1$ by projection onto the first coordinate, where the coordinate on $\mathbb{A}^1$ is t. An element $D \in Div(\mathbb{A}^1)$ is of the form $D = \Sigma n_i p_i$ where p_i is a point in $\mathbb{A}^1$. Then $\pi^*(D) = \Sigma n_i \pi^*(p_i)$. $\underline{0} \in Div(\mathbb{A}^1) \mapsto \pi^*(\underline{0})$, but what is this? Since $\underline{0}$ is defined by $div(t)$, $\pi^*(\underline{0})$ should be defined by $\pi^*(t) = x \in \frac{k[x,y]}{x^2+y^2-1}$.
$\pi^*(div(t)) = div(\pi^*(t)) = div(x)$, since π is defined by $(x, y) \mapsto x$.

$supp(div(x)) = \mathbb{V}(x) \subseteq X = \mathbb{V}(x^2+y^2-1) \Rightarrow supp(div(x)) = \mathbb{V}(x, x^2+y^2-1) = \mathbb{V}(x, (y+1)(y-1))$
In addition, $\pi^*(div(t-1)) = div(\pi^*(t-1)) = div(x-1)$ on X.
$supp(div(x-1)) = \mathbb{V}(x-1) \subseteq X \Rightarrow\Rightarrow supp(div(x-1)) = \mathbb{V}(x-1, x^2+y^2-1) = \mathbb{V}(x-1, y^2) = \{(1,0)\} \equiv Q$

Note that $\frac{k[x,y]}{x^2+y^2-1}(x-1, y) \ni x-1 = \frac{-y^2}{x+1}$, so $\frac{k[x,y]}{x^2+y^2-1}(x-1, y)$ has maximal ideal (y).
Since $x-1$ is generated by $y^2, \nu_Q(x-1) = 2$.

Definition 3.1.43. Let $X \xrightarrow{\varphi} Y$ be a dominant morphism of irreducible varieties. If $D \in CDiv(Y)$ is given by the data $\{U_\lambda, f_\lambda\}$, then the **pull-back of** D, denoted $\varphi^* D$, is given by data $\{\varphi^{-1}(U_\lambda), \varphi^*(f_\lambda)\}$.

Remark 3.1.44. The pull-back of f_λ makes sense because the map $X \xrightarrow{\varphi} Y$ is a dominant morphism so $k(Y) \hookrightarrow k(X)$ and $\mathcal{O}_X(Y) \xhookrightarrow{\varphi^*} \mathcal{O}_Y(V)$ is injective.

Note that $\varphi^* f_\lambda \cdot (\varphi^* f_\mu)^{-1} = \varphi^*(f_\lambda \cdot f_\mu^{-1}) \in \sigma^*_X(\varphi^{-1}(U_\lambda) \cap \varphi^{-1}(U_\mu)) = \varphi^{-1}(U_\lambda \cap U_\mu)$. This follows since $f_\lambda \cdot f_\mu^{-1} \in \mathcal{O}^*_X(U_\lambda \cap U_\mu)$. So we can pull-back Cartier divisors under dominant morphisms. Again D is given by $\{U_\lambda, div_{U_\lambda} f_\lambda\}$ and $\varphi^* D$ is given by $\{\varphi^{-1}(U_\lambda), div_{\varphi^{-1}(U_\lambda)} \varphi^* f_\lambda\}$.

Proposition 3.1.45. *If $X \xrightarrow{\varphi} Y$ is a morphism of irreducible varieties, $D \in CDiv(Y)$ such that $\varphi(X) \nsubseteq suppD$ then we can define $\varphi^*(D)$ exactly as for the dominant case.*

Conclusion: If $X \xrightarrow{\varphi} Y$ is a morphism of irreducible varieities, there is no map of groups $DivY \xrightarrow{\varphi^*} DivX$ in general, but there is a map $PicY \xrightarrow{\varphi^*} PicX$.
CAUTION: One cannot pull-back a Weil Divisors (an element of $DivX$) that is not locally principal in any reasonable way, even on a normal variety.

Lecture 33. Pulling-back Divisors

Pulling Back Cartier Divisors:
Let $X \xrightarrow{\varphi} Y$ be a morphism of irreducible varieties.

Remark 3.1.46. With a normal variety, we can localize at m_p and discuss the "divisors of zeros and poles" of f, since f has an order of vanishing.

Given a locally principal divisor $D \in CDiv(Y)$, we want $\varphi^* D \in CDiv(X)$ ($\varphi^* D$ should take principal divisors to principal divisors).
If D is defined by f_λ on $U_\lambda \subset Y$ open affine, we want $\varphi^* D$ to be defined by $\varphi^* f_\lambda$ on $\varphi^{-1}(U_\lambda) \subseteq X$.

Example 3.1.47. Let $X = \mathbb{V}(x^2+y^2-1) \subset \mathbb{A}^3$ and define $X \xrightarrow{\varphi} \mathbb{A}^2$ by $\varphi : (x, y, z) \mapsto (x, y)$. Given a divisor $D = \Sigma n_i C_i \in CDiv(\mathbb{A}^2) = Div(\mathbb{A}^2)$, we want $\varphi^* D = \Sigma n_i \varphi^*(C_i)$. If $C_i = \mathbb{V}(f_i)$, where f_i is irreducible, then $C_i = div(f_i)$ and so $\varphi^*(C_i) = \varphi^*(div(f)) = div(\varphi^* f_i)$.
Let $L = \mathbb{V}(x) \subseteq \mathbb{A}^2$. Then $\varphi^* L = div_X(\varphi^* x) = div_X x$

$supp(div_X(x)) = \mathbb{V}(x) \subseteq X \Rightarrow supp(div_X(x)) = \mathbb{V}(x) \cap X = \mathbb{V}(x, y^2-1) = \mathbb{V}(x, y-1) \cup \mathbb{V}(x, y+1)$

Let $L_1 = \mathbb{V}(x, y+1)$ and $L_2 = \mathbb{V}(x, y-1)$, so that $div(x) = a_1 L_1 + a_2 L_2$ for some $a_1, a_2 \in k$.
It is easy to see that $a_1 = a_2 = 1$.

Example 3.1.48. Let $L' = \mathbb{V}(y-1)$. Then $\varphi^* L' = \varphi^*(div(y-1)) = div(\varphi^*(y-1)) = div(y-1) = aL_2$.

$supp(div(y-1)) = \mathbb{V}(y-1) \subseteq X = \mathbb{V}(y-1, x^2+y^2-1) \Rightarrow supp(div(y-1)) = \mathbb{V}(y-1, x^2) = L_2$

Therefore $a = 2$. A curve in $\mathbb{A}^2$ pulls back to the line in $\mathbb{A}^3$ over the points where the curve intersects the circle. Let $C = \mathbb{V}(x^2+y^2-1)$. Then $\varphi^* C = \varphi^*(div(x^2+y^2-1)) = div(\varphi^*(x^2+y^2-1)) = div(0)$. Now $Im\varphi \subseteq C$ and so $\varphi^*(f) = f \circ \varphi = 0$. We do not want $Im\varphi$ to be contained in any divisor, since pulling back that divisor would give us something that is undefined. We conclude that in order to define φ^* on all of $CDiv(Y)$, we require that φ^* must be dominant.

Recall: Given $D \in CDiv(Y)$ (locally principle), we define $\varphi^* D \in CDiv(X)$ when $\varphi(X) \nsubseteq SuppD$.
Represent D by $\{U_\lambda, f_\lambda\}$, where $f_\lambda \in k(X)^*$. Without loss of generality we can assume that the U_λ are affine and $f_\lambda = \frac{g_\lambda}{h_\lambda}$, where $g_\lambda, h_\lambda \in \mathcal{O}_X(U_\lambda) \setminus \{0\}$.
Then $\varphi^* D \equiv \{\varphi^{-1}(U_\lambda), \frac{\varphi^* g_\lambda}{\varphi^* h_\lambda}\}$, is well-defined.

Proof. The point is that we need both $\varphi^* g_\lambda$ and $\varphi^* h_\lambda$ to be nonzero.
$Supp(D \cap U_\lambda) = Supp(div f_\lambda)$ on U_λ, which equals $\mathbb{V}(g_\lambda) \cup \mathbb{V}(h_\lambda)$ assuming that g_λ and h_λ do not share any common factors. Assume that $\varphi(X) \nsubseteq SuppD$, so that $\varphi(X) \cap U_\lambda \nsubseteq SuppD \cap U_\lambda = \mathbb{V}(g_\lambda) \cup \mathbb{V}(h_\lambda)$
Then we need $\varphi^* g_\lambda \neq 0$ since if $\varphi^* g_\lambda = 0$, then $\varphi^* g_\lambda = 0$ on $\varphi^{-1}(U_\lambda)$.
There, $\varphi^* g = 0 \Rightarrow g \circ \varphi = 0 \Rightarrow \forall x \in X, g \circ \varphi(x) = 0 \Rightarrow \forall x \in X, \varphi(x) \in \mathbb{V}(g) \Rightarrow Im\varphi \subseteq \mathbb{V}(g)$. This contradicts our assumption. □

Lemma 3.1.49 (Moving Lemma)**.** *Fix a Cartier divisor D in an irreducible variety Y and fix a finite number of points $y_1, \ldots, y_t$. Then there exists a divisor D' such that $D' \sim D$ and $y_i \notin SuppD' \forall i$.*

Proof. We will do the one point case, $y \in Y$, and assume that Y is normal. If D is not effective then $D = D_1 - D_2$ where D_1, D_2 are both effective divisors. Take $D_1 \sim D_1'$ and $D_2 \sim D_2'$ missing y, then set $D' = D_1' - D_2'$. Therefore we can assume that D is effective. We can write D as $\{U_\lambda, f_\lambda\}$, where $f_\lambda \in \mathcal{O}_X(U_\lambda)^*$. Since D is effective, f_λ has no poles. Pick U_λ containing y and set $D' = D - div(f_\lambda)$. Then $D' \sim D$ since $D - D' = div(f_\lambda)$. On $U_\lambda, D \cap U_\lambda = div(f_\lambda)$, so $D' \cap U_\lambda = D \cap U_\lambda - div_{U_\lambda} f_\lambda = 0$. Thus $SuppD' \cap U_\lambda = \varnothing$ so $y \notin SuppD'$. □

Theorem 3.1.50. *If* $X \xrightarrow{\varphi} Y$ *is a morphism of irreducible varieties, there is an induced (functorial) homomorphism:*
$PicY \xrightarrow{\varphi^*} PicX$ *that sends* $[D] \mapsto [\varphi^* D]$*, where* D *is chosen so that* $\varphi(X) \nsubseteq SuppD$*.*

Proof. Fix $[D] \in PicY$. We need to find D' representing $[D]$ such that $Im\varphi \nsubseteq SuppD'$. Choose $y \in Im\varphi$. By the above lemma, we canfind $D' \sim D$ with $y \notin SuppD'$. □

Cool Fact: Say that $X = \mathbb{V}(F_3) \subseteq \mathbb{P}^2$, where F_3 is a smooth curve of degree 3. Then $PicX$ can be identified with an infinitely disjoint union of X, itself indexed by $\mathbb{Z}$, so Picard groups are not always discrete.

3.2. Determining the Relationship between Divisors and Rational Mappings.

3.3. Linear System of Divisors.

Lecture 34.

Definition 3.3.1. A **discrete valuation,** ν**, on a field**, K, is a group map $K^* \longrightarrow \nu\mathbb{Z}$ satisfying $\nu(f+g) \geq min\{\nu(f), \nu(g)\}$.

The corresponding valuation ring is $R_\nu = \{f \in K^* \mid \nu(f) \geq 0\} \cup \{0\} \supseteq m_\nu = \{f \in K^* \mid \nu(f) > 0\} \cup \{0\}$. R_ν is local with maximal ideal m_ν.

Let X be irreducible and normal

Definition 3.3.2. Divisors D, D' on any X are **linearly equivalent**, denoted $D \sim D'$, if they represent the same class in $Cl(X)$. Equivalently, if $\exists f \in k(X)^*$ such that $D = D' + divf$.

Example 3.3.3. Let $X \xrightarrow{\varphi} \mathbb{P}^n$ be a regular map where $\varphi(X) \nsubseteq H = \mathbb{P}^{n-1} \subset \mathbb{P}^n = \mathbb{P}(V)$. We have a linear system of divisors on $\mathbb{P}^n$, namely the hyperplane system $\{H\}_{H \in \mathbb{P}(V^*)}$ Consider the linear system of pull-backs of hyperplanes: $L = \{\varphi^* H\}_{H \in \mathbb{P}(V^*)}$. This is the quintessential linear system. If φ is a closed embedding, then L is the linear system of all hyperplane sections.

Note:

(1) The elements of L are all effective divisors on X, because they are pull-backs by a regular map of effective divisors.
(2) The elements of L are linearly equivalently to each other

Proof. Check: $D_i = \varphi^* H_i \in L$. Say $H_i = \mathbb{V}(l_i)$ where l_i is a linear polynomial. $D_1 - D_2 = \varphi^*(H_1 - H_2) = \varphi^* div(\frac{l_1}{l_2}) = div\big(\frac{\varphi^* l_1}{\varphi^* l_2}\big) = div(\varphi^*\big(\frac{l_1}{l_2}\big))$ □

Basically, a linear system on X will be a collection of effective divisors all linearly equivalent to each other (with one other condition).

Definition 3.3.4. Fix a divisor D on an irreducible, normal variety X. The **Riemann-Roch space** is $\mathcal{L}(D) = \{f \in k(X)^* \mid divf + D \geq 0\} \cup \{0\}$.

Note that there are many different notatons for this.
Examples:

(1) $D = 0, \mathcal{L}(D) = \{f \in k(X)^* \mid divf \geq 0\} \cup \{0\}$ is precisely $\mathcal{O}_X(X)$ because the only globally regular functions on projective space are constant.
(2) Special case: X is projective and $D = 0$, then $\mathcal{L}(D) = k$
(3) There's a map of sets $\mathcal{L}(D) \setminus \{0\} \twoheadrightarrow |D| \subseteq Div(X)$ sending $f \mapsto div(f) + D$ whose image is the set of effective divisors linearly equivalent to D
(4) If X is projective, $\mathcal{L}(D)$ is finite dimensional, and the induced map $\mathbb{P}(\mathcal{L}(D)) \longrightarrow |D|$ is a bijection.
(5) $X = \mathbb{P}^1$, $D = N\{\infty\} = N[0:1], f(t) = f \in k(\mathbb{P}^1)$ where $t = \frac{y}{x}$ on U_x or $f(t) = \frac{F(x,y)}{G(x,y)}$ where F, G are homogeneous polynomials of the same degree.

$$divf + n\{\infty\} \geq 0 \Leftrightarrow \Sigma_p \nu_p(f)p + n\{\infty\} \geq 0$$

When $p \neq \infty, \nu_p(f) \geq 0 \Rightarrow f$ is regular on U_x. So f can be written as a polynomial in t, $f = \frac{F}{G} = \frac{F}{x^t}$, where F is homogeneous of degree t. When $p = \infty$: $\nu_\infty(f) \geq -n$, i.e. f has a pole of order at most n at ∞. So $\frac{F}{x^t}$ can be written $\frac{F}{x^n}$ where F is homogeneous of degree n. Then $\mathcal{L}(D) = \{\frac{F(x,y)}{X^n} \mid F$ homogeneous of degree $n\} \cong Sym^n(V^*)$, as a k-vector space, has dimension $n+1$.

Basic Properties of $\mathcal{L}(D)$, where X is irreducible and normal, D is any divisor.

Proposition 3.3.5. *$\mathcal{L}(D)$ is a k-vector space in $k(X)$ and it is finite dimensional if X is projective.*

Proof. $f \in \mathcal{L}(D) \Rightarrow \lambda f \in \mathcal{L}(D), \lambda \in k^*$ since $div(\lambda f) + D = div(f) + D \geq 0$
$f, g \in \mathcal{L}(D) \Rightarrow f + g \in \mathcal{L}(D)$
$divf + D \geq 0, divg + D \geq 0$ then:

$\nu_Y(f) +$ coef. of $Y in D \geq 0, \nu_Y(g) +$ coef of Y in $D \geq 0 \Rightarrow \nu_Y(f+g) +$ coef of Y in $D \geq 0$

The proof that if X is projective then $\mathcal{L}(D)$ is finite dimensional can be found in Shaf. I (on page 173). □

Proposition 3.3.6. *There is a map of sets from $\mathcal{L}(D) - \{0\} \longrightarrow Div(X)$ that sends $f \mapsto divf + D$ whose image is the set of effective divisors linearly equivalent to D (by definition) - denote this set by $|D|$. Moreover, if X is projective, this identifies the set $|D|$ with $\mathbb{P}(\mathcal{L}(D))$.*

Proof. Need $divf + D = divg + D$ (both effective) $\Leftrightarrow \exists \lambda \in k^*$ such that $f = \lambda g$.
$divf + D = divg + D \Leftrightarrow divf = divg \Leftrightarrow divf - divg = 0 \Leftrightarrow div(\frac{f}{g}) = 0 \Rightarrow \frac{f}{g} = \lambda \neq 0$ □

Lecture 35. Linear Systems

For this lecture, fix X as an irreducible normal variety.

Example 3.3.7. Let $X = \mathbb{P}^2, C = \mathbb{V}(x^2 + y^2 - z^2)$ irreducible cone. Then:
$\mathcal{L}(C) = \{\frac{F(x,y,z)}{G(x,y,z)} \mid divf + C \geq 0\} = \{\frac{F}{(x^2+y^2-z^2)} \mid F$ is homogeneous of degree $2\}$
This is a vector space with basis $\{\frac{x^2}{x^2+y^2-z^2}, \ldots, \frac{yz}{x^2+y^2-z^2}, \frac{z^2}{x^2+y^2-z^2}\}$
There is a one to one correspondence between the following:
$\mathbb{P}(\mathcal{L}(C)) \leftrightarrow \{F \mid F$ is degree 2 over $k^*\} \leftrightarrow \{\mathbb{V}(F) \subseteq \mathbb{P}^2\}_{F \in Sym(\mathbb{P}^2)}$
If instead we used $C' = \mathbb{V}(xy)$ or any $\mathbb{V}(G)$, where G is a homogeneous degree 2 polynomial, we could have:
$\mathcal{L}(C') = \{\frac{F}{G} \mid degF = 2\} \cong \mathcal{L}(C)$.

Proposition 3.3.8. *If $D \sim D'$, then $\mathcal{L}(D) \cong \mathcal{L}(D')$ as a k-vector space.*

Proof. Say $D - D' = divg$ for some $g \in k(X)^*$. $\mathcal{L}(D) \longrightarrow \mathcal{L}(D')$, multiplication by g, sends $f \mapsto gf$. The inverse of this map is multiplication by g^{-1}. Check: $divf + D \geq 0 \Rightarrow div(gf) + D' \geq 0$. □

Definition 3.3.9. A **complete linear system**, $|D|$, is a complete set of all effective divisors linearly equivalent to D. Equivalently, $|D|$ is the image of $\mathcal{L}(D) \xrightarrow{\psi} Div(X)$, where $f \mapsto div(f) + D$.

Definition 3.3.10. A **linear system** on X is a collection of effective linearly equivalent divisors corresponding to some vector subspace of $\mathcal{L}(D)$.

Example 3.3.11. On $\mathbb{P}^n$, $|H| = \{H\}_{H\in(\mathbb{P}^n)^*}$. Fix $H_0 = \mathbb{V}(x_0)$ so that $\mathcal{L}(H_0) = \{\frac{l}{x_0} \mid l$ homogeneous of degree $1\} \subseteq k(\mathbb{P}^n)$. This has basis $\{\frac{x_0}{x_0}, \dots, \frac{x_n}{x_0}\}$. Consider the vector subspace, W, spanned by $\{\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\}$.

$$\mathcal{L}(H_0) \twoheadrightarrow |H_0| \text{ by sending } \frac{l}{x_0} \mapsto div(\frac{l}{x_0}) + H_0 = \mathbb{V}(l)$$

An element $\frac{a_1x_1+\dots+a_nx_n}{x_0} \in W$ is sent to
$div(\frac{a_1x_1+\dots+a_nx_n}{x_0}) + H_0 = \mathbb{V}(a_1x_1 + \dots + a_nx_n)$, the hyperplane through $p = [1 : 0 \cdots : 0]$.
This is a quintessential example of a (non-complete) linear system:hyperplane in $\mathbb{P}^n$ passing through p.

Connection between linear systems and maps to $\mathbb{P}^n$
Fix a linear system $|W|$ on X, so $|W|$ = image of W (this is a subspace) $\subseteq \mathcal{L}(D)$ under the map ψ.

Fix a basis $\varphi_0, \dots, \varphi_n$ for W in $\mathcal{L}(D)$. This gives a map $X \overset{\varphi_{|W|}}{\dashrightarrow} \mathbb{P}^n$ that sends $x \mapsto [\varphi_0(x) : \dots : \varphi_n(X)]$. This is the rational map associated to W.

Example 3.3.12. Let $|W|$ = Hyperplanes in $\mathbb{P}^n$ through $p = [1 : 0 : \dots : 0]$ (note that the standard notation for W is $|H_p|$). $|W|$ has a basis $\frac{x_0}{x_0}, \dots, \frac{x_n}{x_0}$ ($|W|$ inside $\mathcal{L}(H_0)$ where $H_0 = \mathbb{V}(x_0)$)
$\mathbb{P}^n \overset{\varphi_{|W|}}{\dashrightarrow} \mathbb{P}^{n-1}$ sending $[x_0 : \dots : x_n] \mapsto [x_1 : \dots : x_n]$ is projection from p!

Example 3.3.13. $|C|$ on $\mathbb{P}^2$, $\mathbb{P}^2 \overset{\varphi_{|C|}}{\dashrightarrow} \mathbb{P}^5$ sends $[x : y : z] \mapsto [x^2 : \dots : z^2]$ is the Veronese map ν_2.

Example 3.3.14. Let $W = span\{\frac{x^2}{G}, \frac{xy}{G}\} \subseteq \mathcal{L}(C)$. Then we have a map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ where $[x : y : z] \mapsto [\frac{x^2}{G} : \frac{xy}{G}] = [x : y]$. So $|W| = \{div(\frac{ax^2+bxy}{G}) + \mathbb{V}(G)\} = \mathbb{V}(x) \cup \mathbb{V}(ax + by)$ in $\mathbb{P}^2$.

Definition 3.3.15. A prime divisor Y is a **fixed component** of a linear system $|W|$ if $Y \subseteq Supp(D) \forall D \in |W|$.

Remark 3.3.16. If $|W|$ has a fixed component Y, then $|W|$ and $|W - Y| = \{D - Y \mid D \in |W|\}$ determine the same map, $\varphi_{|W|}$ and $\varphi_{|W-Y|}$, to $\mathbb{P}^n$ (up to a linear change of coordinates in the target projective space).

Lecture 36.

Theorem 3.3.17. *If X is normal and irreducible over k, then there is a one-to-one correspondence:*

$\{$Non-degen. rational maps $X \overset{\varphi}{\dashrightarrow} \mathbb{P}^n\} \leftrightarrow \{$n-dim. linear systems of divisors with no fixed components$\}$ Where $\varphi \mapsto \{\varphi^*(|H|\}_{H\in(\mathbb{P}^n)^*}$. Furthermore, the locus of indeterminacy of $\varphi = \varphi_{|W|}$ is precisely $\cap_{D\in|W|} SuppD = \cap_{H\in(\mathbb{P}^n)*} Supp(\varphi^*H)$.

Proof. (Sketch) Let $Z \subseteq X$ be the locus of indeterminacy, so Z is closed of codimension at least 2. Let $U = X \backslash Z$. In general, $DivX \xrightarrow{\cong} DivU$ sending $D = \Sigma m_iY_i \mapsto D \cap U = \Sigma m_i(Y_i \cap U)$ and $\overline{D} = \Sigma m_i\overline{Y_i} \longleftarrow \Sigma m_iY_i = D$.
It is easy to check that:

(1) Given $\varphi : X \dashrightarrow \mathbb{P}^n$, we get $|W| = \{\varphi^*(H)\}_{H\in(\mathbb{P}^n)^*}$ and $\varphi_{|W|} = \varphi$

(2) Given $|W|$, choose D basis $f_0, \dots, f_n$ for $W \subseteq \mathcal{L}(D)$ and get a map $X \overset{\varphi_{|W|}}{\dashrightarrow} \mathbb{P}^n$ where
$x \mapsto [f_0 : \cdots : f_n]$. and $\{\varphi^*_{|W|}(|H|)\} = |W|$.

Check the statement about the locus of indeterminacy and refer to Shaf III §1.5 for the other direction. □

Definition 3.3.18. The **base locus** of a linear system $|W|$, denoted $Bs|W|$, is the intersection of $SuppD$ for $D \in |W|$. $Bs|W| = \cap_{D\in|W|} SuppD$. Equivalently, $Bs|W|$ = the locus of indeterminacy of $\varphi_{|W|}$ (if $|W|$ has no fixed components.)

Definition 3.3.19. A **base point free** linear system $|W|$ is one where the base locus is empty (i.e. $\varphi_{|W|}$ is regular).

Definition 3.3.20. A **very ample** linear system $|W|$ is one where $\varphi_{|W|}$ is an embedding.

Definition 3.3.21. A divisor D is **ample** if $\exists n > 0$ such that $|nD|$ is very ample.

Example 3.3.22. If $|C|$ = linear system of all conics on $\mathbb{P}^2$, then $Bs|C| = \varnothing$. This is given by the Veronese map $\nu_2 : \mathbb{P}^2 \longrightarrow \mathbb{P}^5$. This is a base point free linear system and very ample.

Example 3.3.23. $|H_p|$ = linear system of hyperplanes through the point p. Then $Bs|H_p| = \{p\}$. This is not base point free nor very ample.

Philosophical Problem: We need a canonical way to find a linear system on a (normal projective) variety X. We can do this by using differential forms.
Fix a variety X over k. At each point $p \in X$, we have a tangent space (i.e. k-vector space) T_pX and a cotangent space $(T_pX)^*$

Definition 3.3.24. A (completely arbitrary) differential form is an assignment $\forall p \in X, \omega_p \in (T_pX)^*$.

Example 3.3.25. Let $X = \mathbb{A}^n$ and $f \in k[\mathbb{A}^n] = k[x_1, \dots, x_n]$ and df is a differential form that assigns to $p = (\lambda_1, \dots, \lambda_n) \in \mathbb{A}^n$, the linear functional $d_pf = \Sigma_{i=1}^n \frac{\partial f}{\partial x_i}|_p(x_i - \lambda_i)$ linear function on $T_p\mathbb{A}^n$. In particular, dx is a differential form at $p = (\lambda_1, \dots, \lambda_n) \rightsquigarrow d_px_i = x_i - \lambda_i$. So $dx_1, \dots, dx_n$ are a basis for $(T_p\mathbb{A}^n)^*$ at each point p. So, equivalently, $df = \Sigma_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$. Note that w_1, w_2 are differental forms, then so are $w_1 + w_2$ and fw_1, where f is any function in $k[x_1, \dots, x_n]$. In particular, $\Omega_X(U) = \{$ differential forms on $U\}$ is a module over $\mathcal{O}_X(U)$.

Example 3.3.26. (Quintessential) On $\mathbb{A}^n$, every regular (i.e. polynomial) differential form can be written uniquely, as $g_1dx_1 + \cdots + g_ndx_n$, where the g_i are regular on $\mathbb{A}^n$.

Definition 3.3.27. A **regular differential form** on a variety X over k is a differential form w with the property that there exists an open affine cover $\{U_\lambda\}$ of X such that $w|_{U_\lambda}$ agrees with the differential form $g_1df_1 + \dots g_tdf_t$ for some $g_i, f_j \in \mathcal{O}_X(U_\lambda)$.

Definition 3.3.28. If X is a smooth projective curve the **genus** of X is $dim_x(\Omega_x(X))$.

Lecture 37. Differential Forms

If M is a free R-module of rank n, then Λ^pM is a free R-module of rank $\binom{n}{p}$.
If $u_1, \dots, u_n$ is a free basis for M, then $\{u_{i_1} \wedge \cdots \wedge u_{i_p}\}_{i_1<i_2<\cdots<i_p}$ is a free basis for Λ^pM.
If $u`_1, \dots, u`_n$ is a different basis, with $u` = \Sigma_{j=1}^n h_{ij}u_j$ (i.e. (h_{ij}) is the change of basis matrix), then
$u`_1 \wedge \cdots \wedge u`_n = det(h_{ij})u_1 \wedge \cdots \wedge u_n$.

Let X be an irreducible variety over k
A differential form on X is an assignent ω taking each point p to $\omega_p \in (T_pX)^*$

If X is affine we say that ω is a regular differential form if it agrees with $\Sigma_{i=1}^{t} f_i dg_i$, where $f_i, g_i \in \mathcal{O}_x(X)$.

Remark 3.3.29. If $X \subseteq \mathbb{A}^n$ is a closed set and $f_i, g_i \in k[X] = \frac{k[x_1,\dots,x_n]}{\mathbb{I}(X)}$. Each $dg = \Sigma_{i=1}^{n} \frac{\partial g}{\partial x_i} dx_i$,
so $\omega = \Sigma_{i=1}^{n} h_i dx_i$, where $h_i \in k[X]$.

Definition 3.3.30. For $U \subseteq X$ open, $\Omega_X(U) = \{\omega \mid \omega \text{ is a regular differential form on } U\}$. So $\Omega_X(U)$ is a $\Omega_X(U)$-module. Ω_X is a sheaf of Ω_X-modules.

Remark 3.3.31. Unless stated otherwise, our differential forms are assumed regular since that's our category.

We have an additive map: $\mathcal{O}_X \xrightarrow{d} \Omega_X$ where $f \mapsto df$.

Example 3.3.32. Consider the global differential forms on $\mathbb{A}^n$, $\Omega_{\mathbb{A}^n}(\mathbb{A}^n) =$ free $k[x_1,\dots,x_n]$-module on $dx_1,\dots,dx_n$.

Theorem 3.3.33. *Let X be a smooth variety of dimension n. Then Ω_X is a locally free sheaf of $\mathcal{O}_X$ modules of dimension n (since the cotangent space is of dimension n). In particular, if $u_1,\dots,u_n$ are parameters at p, then $du_1,\dots,du_n$ are a free basis for Ω_X in a neighborhood of p.*

Proof. Can be found in Shaf. I, in the section on differentials. "If this were a year long course, I would prove this for you. But the proof is not too hard. Just follow your nose." ☐

Example 3.3.34. Let $X = \mathbb{P}^1$ have coordinates x, y. Since x, y are not regular functions, we cannot write dx, dy. Let $s = \frac{y}{x}$ on U_x and $t = \frac{x}{y}$ on U_y so that on the intersection, $t = \frac{1}{s}$.

$$\Omega_X(U_x) \cong \Omega_{\mathbb{A}^1}(\mathbb{A}^1) = k[t]dt$$

$$\Omega_X(U_y) \cong \Omega_{\mathbb{A}^1}(\mathbb{A}^1) = k[s]ds$$

Say $\omega \in \Omega_{\mathbb{P}^1}(\mathbb{P}^1)$. Write $\omega = P(t)dt$ in U_x and $\omega = Q(s)ds$ in U_y. On $U_x \cap U_y$,
$dt = d(\frac{1}{s}) = \frac{-ds}{s^2} \Rightarrow P(t)dt = P(\frac{1}{s})\frac{-ds}{s^2} = Q(s)ds \Rightarrow Q(s) = P(\frac{1}{s})\frac{-1}{s^2} \Rightarrow -s^2Q(s) = P(\frac{1}{s}) \Rightarrow$
$P = Q = 0$. Therefore there are no global regular differential forms on $\mathbb{P}^1$.

Example 3.3.35. Let $X = \mathbb{V}(x^3 + y^3 + z^3) \subseteq \mathbb{P}^2$, which is a smooth curve. Let $U_1 = X \cap U_x$ and $U_2 = X \cap U_y$.
On $U_1, X = \mathbb{V}(1 + (\frac{y}{x})^3 + (\frac{z}{x})^3) \subseteq \mathbb{A}^2$, let $s = \frac{y}{x}$ and $t = \frac{z}{x}$.
Then $\Omega_X(U_1)$ is generated by ds, dt as an $\Omega_X(U_1) = \frac{k[s,t]}{s^3+t^3+1}$ since:
$s^3 + t^3 + 1 = 0 \Rightarrow 3s^2ds + 3t^2dt = 0 \Rightarrow ds = \frac{-t^2}{s^2}dt$ on $U_1 \cap U_y$ and $dt = \frac{-s^2}{t^2}ds$ on $U_1 \cap U_z$
So $\Omega_X(U_1 \cap U_y) = \mathcal{O}_X(U_1 \cap U_y)dt$ and $\Omega_X(U_1 \cap U_z) = \mathcal{O}_X(U_1 \cap U_z)ds$.

Definition 3.3.36. A **differential p-form** on X is an assignment ω to each $q \in X$, $\omega_q \in \Lambda^p((T_qX)^*)$.

Definition 3.3.37. A **regular differential p-form** is an assigment ω which, locally on affine charts, looks like $\Sigma g_{i_1,\dots,i_p} df_{i_1} \wedge df_{i_2} \wedge \cdots \wedge df_{i_p}$ where $f_i, g_{i_1,\dots,i_p}$ are regular functions.

For $U \subseteq X$ open, $\Omega_X^p(U)$ denotes the set of all regular p-forms on $U, \Omega_X(U)$-module. So Ω_X^p is a sheaf of Ω_X-modules.

Corollary 3.3.38. *If X is smooth, then Ω_X^p is a locally free $\mathcal{O}_X$-module of rank $\binom{n}{p}$.*

The most important of these is Ω_X^n, locally free of rank 1, called the canonical module. It is also denoted by ω_X.

Rational Differential Forms:

Loosely, a rational differential form is a regular differential form ω on some non-empty open set. Precisely,

Definition 3.3.39. A **rational differential p-form** is an equivalence class (U,ω) where $U \subseteq X$ is a non-empty open set and ω is a regular p-form on U. Then $(U,\omega) \sim (U`,\omega`) \Leftrightarrow \omega|_{U\cap U`} = \omega`|_{U\cap U`}$.

Just as we defined before, $div(f)$ for $f \in k(X)^*$, we will define $div(\omega)$ for ω rational n-form. The different $div(\omega)$ as ω ranges through all rational differential n-forms on the smooth variety X of dimension n produce linearly equivalent divisors, that class they give is K_X the canonical class. So there is a special divisor class (besides 0) in the divisor class group. So any element in K_X can be written as $div\omega$ for some rational differential n-form on X, when $dimX = n$.

Lecture 38.

Let X be an irreducible, smooth (this is optional, but easier to imagine) variety over k of dimension n

Ω_X is a sheaf of regular differential forms of locally free $\mathcal{O}_X$-modules of rank n

ω_X is the canonical sheaf (n-forms, $\Lambda^n(\Omega_X)$), locally free $\mathcal{O}_X$-modules of rank 1

Fix a point p and parameters $u_1,\ldots,u_n$ parameters at p

$du_1,\ldots,du_n$ are generators of $\mathcal{O}_X$-modules in a neighborhood of p

$du_1 \wedge \cdots \wedge du - n$ are generators of $\mathcal{O}_X$-module in a neighborhood of p

Proposition 3.3.40. *Fix any non-empty open $U \subseteq X$ such that $\omega_X(u) = \mathcal{O}_X(U)du_1 \wedge \cdots \wedge du_n$ for some choice of $u_1,\ldots,u_n$. Given any rational n-form ω, there is a unique representation as $\omega = gdu_1 \wedge \cdots \wedge du_n$, where $g \in k(X)$. In particular, the set of rational n-forms on X forms a 1-dimensional $k(X)$-vector space.*

Proof. Take any rational n-form, say represented by (U',ω). Instead we can represent it by $(U\cap U',\omega)$ and $\omega = gdu_1 \wedge \cdots \wedge du_n$ so g is regular on $U \cap U'$ and $g \in k(X)$. □

Example 3.3.41. $\mathbb{P}^1$ with coordinates x,y and $t = \frac{y}{x}, s = \frac{x}{y}$. On $U_x \cong \mathbb{A}^1$, $dt_p = (t-\lambda)$, where $p = (\lambda)$, is a linear function on $T_p\mathbb{A}^1 = \mathbb{A}^1$ with the origin located at p. On U_y, $dt = -\frac{ds}{s^2}$, which has a pole of order 2. So $div(dt) = -2$ at ∞.

Divisor of a rational differeintial n-form on the smooth n-dimensional variety X:

First: Fix an open cover of X, say $\{U_\lambda\}$ such that $\omega_X(U_\lambda) = \mathcal{O}_X(U_\lambda)du_1^\lambda \wedge \cdots \wedge du_n^\lambda$. Given a rational form ω, represented as $\omega = g_\lambda du_1^\lambda \wedge \cdots \wedge du_n^\lambda$ on U_λ. Consider the data $\{(U_\lambda, g_\lambda)\}_\lambda$, where $g \in k(X)$. This defines a well-defined divisor on X.

Note: This is a well-defined divisor because $g_\lambda \cdot g_\mu^{-1} \in \mathcal{O}_X^*(U_\lambda \cap U_\mu)$. This holds since:

$$\omega = g_\lambda du_1^\lambda \wedge \cdots \wedge du_n^\lambda = g_\mu du_1^\mu \wedge \cdots \wedge du_n^\mu$$

Also, $du_1^\lambda \wedge \cdots \wedge du_n^\lambda$ and $du_1^\mu \wedge \cdots \wedge du_n^\mu$ generate $\omega_X(U_\lambda \cap U_\mu) \cong \mathcal{O}_X(U_\lambda \cap U_\mu)$. There exists an invertible element $h \in \mathcal{O}_X(U_\lambda \cap U_\mu)$ such that $du_1^\lambda \wedge \cdots \wedge du_n^\lambda = hdu_1^\mu \wedge \cdots \wedge du_n^\mu$.

Definition 3.3.42. A **canonical divisor** is any divisor of the form $div(\omega)$, where ω is a rational n-form on X.

Proposition 3.3.43. *All canonical divisors form a linear equivalence class.*

Proof. Since the set of all such rational forms is 1-dimensional over $k(x)$. Fix any non-zero ω, any other will be of the form $g\omega$ for some $g \in k(X)$. □

Example 3.3.44. Compute the canonical class of $\mathbb{P}^2$, with coordiates $x : y : z$. Let $s = \frac{x}{z}, t = \frac{y}{z}$ which are rational functions on U_z. Choose $\omega = ds \wedge dt$. On U_z, $(g_\lambda = 1)$ so $div(\omega) \cap U_z = 0$. Let $L = \mathbb{V}(z)$. Then $div(\omega) = dL$ for some d. Let's find d: On U_y, we have coordinates $u = \frac{x}{y}, v = \frac{z}{y}$ so that $t = \frac{1}{v}, s = \frac{u}{v}$.

$$\begin{aligned} ds \wedge dt &= d(\frac{u}{v}) \wedge d(\frac{1}{v}) \\ &= (\frac{vdu - udv}{v^2}) \wedge (\frac{-dv}{v^2}) \\ &= \frac{-vdu \wedge dv}{v^4} \\ &= \frac{-du \wedge dv}{v^3} \end{aligned}$$

$div(\omega) \cap U_y = -3\mathbb{V}(v) = -3L \Rightarrow div(\omega) = -3L$.
On $\mathbb{P}^n$ the canonical class is $-(n+1)H$

Exercise: Let $X \subseteq \mathbb{P}^n$ be a smooth hypersurface of degree d, then $K_X = (d - n - 1)H_X$, where H_X is a hypersurface class on X $(\{H \cap X\}_{H \in (\mathbb{P}^n)^*})$
Recall: There is a bijection between:
{ rational maps to projective space, up to a linear change of coordinates} and { linear systems of divisors }

$|W| \subseteq |D|, W \subseteq \mathcal{L}(D)$. $\mathcal{O}_X(D)(U) = \{f \in k(X) \mid div f + D \geq 0 \text{ on } U\}$ is a locally free sheaf.

Definition 3.3.45. A **line bundle** on X is a surjective regular map $L \xrightarrow{\pi} X$ with the property that X has an open cover U_λ such that: if $\pi^{-1}(U_\lambda) \xrightarrow[\cong]{\varphi_\lambda} U_\lambda \times k$ and $\pi^{-1}(U_\mu) \xrightarrow[\cong]{\varphi_\mu} U_\mu \times k$, then

$$\begin{array}{ccc} \pi^{-1}(U_\lambda \cap U_\mu) & \xrightarrow[\cong]{\varphi_\lambda} & (U_\lambda \cap I_\mu) \times k \\ & \searrow^{\varphi_\mu} & \downarrow{\scriptstyle \varphi_\mu \cdot \varphi_\lambda^{-1}} \\ & & (U_\lambda \cap U_\mu) \times k \end{array}$$

where $\varphi_\mu \cdot \varphi_\lambda^{-1} : (x, \lambda) \mapsto (x, g(x)\lambda)$ for $g \in \mathcal{O}_X^*(U_\lambda \cap U_\mu)$

Definition 3.3.46. A **section** of a line bundle $L \xrightarrow{\pi} X$ over $U \subseteq X$ is a (regular) map $U \xrightarrow{s} L$ such that $\pi \circ s = id|_U$. These form a sheaf $\mathcal{L}(U) =$ sections on U.

{line bundles on X} and {locally free rank 1 $\mathcal{O}_X$-modules}/ $\cong$ correspond bijectively by the map:
$\{L \xrightarrow{\pi} X\} \longrightarrow \mathcal{L}$

Lecture 39. Questions Yet to be Answered

Big Questions:
What kinds of varieties are there and how do they relate to each other?

(1) Classify varieties up to birational equivalence
(2) Classify up to isomorphism, find moduli space whose points are in one-to-one corespondence with isomporphism classes of varieties.
(3) Find a parametric space whose point parametrizes the closed subschemes of $\mathbb{P}^N$

Definition 3.3.47. Fix a closed subvariety (or subscheme) $X \subseteq \mathbb{P}^n$. $\mathbb{I}(X) \subseteq kx_0, \ldots, x_n \rightsquigarrow S = \frac{kx_0,\ldots,x_n}{\mathbb{I}(X)}$ is a "homogeneous coordinate ring of $X \subseteq \mathbb{P}^n$" and it is an $\mathbb{N}$-graded ring. The **Hilbert function** $t \mapsto dim_k[S]_t$ = elements of S of pure degree t.

Theorem 3.3.48. *For $t >> 0$, this agrees with a polynomial $P_X(t)$, called the Hilbert polynomial.*

Example 3.3.49. $X = \mathbb{P}^n, S = kx_0, \ldots, x_n$ then $P_{\mathbb{P}^n}(t) = \binom{n+t}{n} = \frac{t^n}{n!} +$ loc t.

Example 3.3.50. $\mathbb{V}(F_d) \subseteq \mathbb{P}^n, S = \frac{kx_0,\ldots,x_n}{(F_d)}$. We can determine $[S]_t$ by looking at the exact sequence:
$[S]_{t-d} \xrightarrow{F_d} [S]_t \longrightarrow [S/(F_d)]_t \longrightarrow 0$ so $dim[S/(F_d)]_t = dim[S]_t - dim[S]_{t-d} = \binom{n+t}{n} - \binom{n+t-d}{n}$
So $P_X(t) = \frac{dt^{n-1}}{(n-1)!} + loct.$

Remark 3.3.51. The leading degree of $P_X(t)$ equals the dimension of X.

Theorem 3.3.52. *Fix a Hilbert polynomial $P(t)$, then there is a scheme over k, $\mathbb{H}^p$ whose closed points are in one-to-one correspondence with the set of closed subschemes of $\mathbb{P}^n$ with Hilbert polynomial $P(t)$.*

Proof. The proof of this is long and complicated - good summer reading. □

Example 3.3.53. Let $P(t) = \binom{n+t}{n} - \binom{n+t-d}{n}$. Then $\mathbb{H}_p$ is the set of hypersurfaces in $\mathbb{P}^n$ of degree d, which is $\mathbb{P}(Sym^d(k^{n+1})^*)$.

Classification of Curves:
In this case, birational equivalence is the same as an isomorphism for smooth projective curves. There is a unique smooth projective representative for each birational class.

Definition 3.3.54. The **genus** of X, denoted by $g(x)$, is the dimension over k of the space of global differential forms.

Example 3.3.55. Fix a space X and find the genus, g, of X:

- $g = 0 \Leftrightarrow \mathbb{P}^1$
- $g = 1 \Leftrightarrow X \subseteq \mathbb{P}^2$ (embedded), $X = \mathbb{V}(F_3)$ are elliptic curves
- $g = 2 \Leftrightarrow \mathcal{L}(K_X)$ has basis φ_0, φ_1, so $X \xrightarrow{\varphi_{|K_X|}} \mathbb{P}^1$ sending $x \mapsto [\varphi_0(x) : \varphi_1(x)]$. A genus 2 curve admits a $2:1$ cover of $\mathbb{P}^1$.
- $g \geq 3$, $\mathcal{L}(K_X)$ has dimension g: $\varphi_0, \ldots, \varphi_{g-1}$ is a basis for $\mathcal{L}(X)$ and determines the map:
  $X \xrightarrow{\varphi_{k_X}} \mathbb{P}^g$, There are two possibilities:
  (1) φ_{K_X} is an embedding
  (2) φ_{K_X} is not an embedding. In this case, the images is isomorphic to $\mathbb{P}^1$ and the map is 2:1. This is called the hyperelliptic curve.

Construct the moduli space of smooth projective curves of genus $g \geq 2$.

Theorem 3.3.56. *$2K_X$ is very ample when $g \geq 2$. The formula for the dimension of $\mathcal{L}(D)$ is given by the Riemann-Roch formula.*

Corollary 3.3.57. *Every curve of genus $g \geq 2$ embeds $X \xhookrightarrow{\varphi_{2K_X}} \mathbb{P}^{3g-4}$ of degree $(2g-2)$.*

(1) To build a parametric space of curves of genus g, look at the Hilbert scheme $\mathbb{H}^{(2g-2)t-g}$
(2) Linear change of coordinates, $PGL(3g-3)$ acts on $\mathbb{H}^p$. We need to construct a quotient variety $\mathbb{H}^p/PGL(3g-3) = M_g$.

4. Glossary

Definition 4.0.58. An **affine algebraic set** is the common zero set of a collection of polynomials $\{F_\lambda\}_{\lambda\in\Lambda}$, where $F_\lambda \in k[x_1, ..., x_n]$ and k is any field. This is denoted by $\mathbb{V}(\{F_\lambda\}_{\lambda\in\Lambda})$.

Definition 4.0.59. Let $k \hookrightarrow L$ be a field extension. Then $x \in L$ is **algebraic over k** if it satisfies $x^n + a_1x^{n-1} + \ldots + a_n = 0$ where $a_i \in k$. Otherwise k is **transcendental**

Definition 4.0.60. Given a projective algebraic variety $V = \mathbb{V}(\{F_\lambda\}_{\lambda\in\Lambda}) \subseteq \mathbb{P}^n_k$, the **affine cone** over V is the affine algebraic set in $\mathbb{A}^{n+1}$ defined by the same polynomials.

Definition 4.0.61. Let k be a field. **Affine n-space**, $\mathbb{A}^n_k$, is a vector space of dimension n over k.

Definition 4.0.62. An **algebraic variety** is a geometric object that looks locally like the zero set of a collection of polynomials.

Definition 4.0.63. Elements $x_1, \ldots, x_d \in L$ are **algebraically independent over k** if they satisfy no (non-zero) polynomial $F(u_1, \ldots, u_d) \in k[u_1, \ldots, u_d]$

Definition 4.0.64. A divisor D is **ample** if $\exists n > 0$ such that $|nD|$ is very ample.

Definition 4.0.65. The **base locus** of a linear system $|W|$, denoted $Bs|W|$, is the intersection of $SuppD$ for $D \in |W|$. $Bs|W| = \cap_{D\in|W|} SuppD$. Equivalently, $Bs|W| =$ the locus of indeterminacy of $\varphi_{|W|}$ (if $|W|$ has no fixed components.)

Definition 4.0.66. A **base point free** linear system $|W|$ is one where the base locus is empty (i.e. $\varphi_{|W|}$ is regular).

Definition 4.0.67. A polynomial, $F(x_0, \ldots, x_n, y_0, \ldots, y_m)$ is **bihomogeneous** if it is homogeneous in the x_i's and homogeneous in the y_i's.

Definition 4.0.68. Irreducible varieties V and W are **birationally equivalent**, denoted $V \sim W$, if there are dominant rational maps $V \overset{F}{\dashrightarrow} W$ and $W \overset{G}{\dashrightarrow} V$ such that $F \circ G$ and $G \circ F$ are the identity rational maps on W and V, respectively.

Definition 4.0.69. The **blow-up** of $\mathbb{A}^2$ along $p \in \mathbb{A}^2$ is the projection morphism $B_p \xrightarrow{\pi} A^2$, where $\mathbb{P}^1$ is the set of lines through p in $\mathbb{A}^2$.

Definition 4.0.70. Let V be an affine variety, $W \subseteq V$ a closed subvariety. Say $\mathbb{I}(W) \subseteq k[V]$ has generators $F_0, \ldots, F_t$. The **blow-up of V along W**, $B_W V$, is the graph of the rational map $V \dashrightarrow \mathbb{P}^t$ sending $\overline{x} \mapsto [F_0(\overline{x}) : \cdots : F_t(\overline{x})]$, together with projection onto the first coordinate.

Definition 4.0.71. The **blow-up of an affine variety, V, along an ideal** $I = (F_0, \ldots, F_t)$, denoted by $B_I V$, is the graph of the rational map $V \dashrightarrow \mathbb{P}^t$ sending $x \mapsto [F_0(x) : \ldots : F_t(x)]$ together with projection onto V.

Definition 4.0.72. A **canonical divisor** is any divisor of the form $div(\omega)$, where ω is a rational n-form on X.

Definition 4.0.73. A **Cartier divisor** D on X is equivalent to the data $\{U_\lambda, f_\lambda\}$ where:

(1) $\{U_\lambda\}$ is an open cover of X
(2) $f_\lambda \in k(X)^*$
(3) $\frac{f_\lambda}{f_\eta} \in \mathcal{O}^*_X(U_\lambda \cap U_\eta)$, invertible regular functons on $U_\lambda \cap U_\eta$

Definition 4.0.74. A **complete linear system**, $|D|$, is a complete set of all effective divisors linearly equivalent to D. Equivalently, $|D|$ is the image of $\mathcal{L}(D) \xrightarrow{\psi} Div(X)$, where $f \mapsto div(f) + D$.

Definition 4.0.75. A **conic** is a degree two hypersurface in $\mathbb{P}^2$.

Definition 4.0.76. The **coordinate ring of** V, $k[V]$, is the set of regular functions in V, considered with the obvious pointwise addition and multiplication of functions.

Definition 4.0.77. If $F \in k[x_0, \ldots, x_n]$ is homogeneous of degree t, we **de-homogenize** (with respect to x_0) F by setting $f(x_1, \ldots, x_n) = F(1, x_1, \ldots, x_n)$.

Definition 4.0.78. A **differential p-form** on X is an assignment ω to each $q \in X$, $\omega_q \in \Lambda^p((T_qX)^*)$.

Definition 4.0.79. The **dimension** of an irreducible quasi-projective variety V is the transcendence degree of $k(V)$ over k.

Definition 4.0.80. The **dimension** of a (non-irreducible) quasi-projective variety is the maximal dimension of its irreducible components

Definition 4.0.81. A **discrete valuation, ν, on a field**, K, is a group map $K^* \longrightarrow \nu\mathbb{Z}$ satisfying $\nu(f+g) \geq min\{\nu(f), \nu(g)\}$.

Definition 4.0.82. Such a domain is called a **discrete valuation ring** (DVR).

Definition 4.0.83. A **divisor** D on X is a finite formal $\mathbb{Z}$-linear combination of prime divisors: $D = \Sigma_i n_i Y_i$, where $n_i \in \mathbb{Z}$ and Y_i is irreducible of codimension 1 in X.

Definition 4.0.84. $Div(X)$ is the free abelian group generated by prime divisors on X

Definition 4.0.85. The **divisor class group** of (an irreducible variety) X, $Cl(X)$, is the group $Div(X)/P(X)$).

Definition 4.0.86. The locus of points where $\varphi \in k(V)$ is regular is the **domain of definition** of φ.

Definition 4.0.87. $X \xrightarrow{\varphi} Y$ is **dominant** if $\varphi(X)$ is dense in Y.

Definition 4.0.88. A divisor $D = \Sigma_i n_i Y_i$ of X is **effective** if all of the $n_i \geq 0$.

Definition 4.0.89. The **embedding dimension** of V at p is the dimension of m/m^2.

Definition 4.0.90. Let $R \hookrightarrow S$ be an (injective) extension of rings. An element $s \in S$ is **integral over** R if it satisfies a monic polynomial with coefficients in R

Definition 4.0.91. A **family of varieties** is a surjective morphism $X \xrightarrow{\pi} B$ of varieties. The fibers are the **members** of the family. The base, B, **parametrizes** the members of the family, $\{\pi^{-1}(b)\}_{b \in B}$

Definition 4.0.92. A morphism of affine varieties, $X \xrightarrow{\varphi} Y$, is **finite** if it is dominant and the corresponding map of coordinate rings $k[Y] \xrightarrow{\varphi^*} k[X]$ is integral.

Definition 4.0.93. A morphism of quasi-projective varieties, $X \xrightarrow{\varphi} Y$, is **finite** if φ is dominant and $\forall y \in Y$ there is an open affine neighborhood U of y such that $\varphi^{-1}(U)$ is affine and $\varphi^{-1}(U) \xrightarrow{\varphi} U$ is finite (equivalently, $\mathcal{O}_Y(U) \xrightarrow{\varphi^*} \mathcal{O}_X(\varphi^{-1}(U))$ is integral).

Definition 4.0.94. A prime divisor Y is a **fixed component** of a linear system $|W|$ if $Y \subseteq Supp(D) \forall D \in |W|$.

Definition 4.0.95. A family $\mathbb{X} \xrightarrow{\pi} B$ is **flat** if there is an affine cover $\{U_i\}$ of B and an affine cover $\{V_{i,j}\}$ of each $\pi^{-1}(U_i)$ such that the induced map of affine varieties $V_{i,j} \xrightarrow{\pi|_{V_{i,j}}} U_i$ induces a flat map of algebras $\mathcal{O}_B(U_i) \longrightarrow \mathcal{O}_{\mathbb{X}}(V_{i,j})$.

Definition 4.0.96. $A \xrightarrow{f} B$ is a **flat map of rings** if for all short exact sequences of A-modules, $0 \to M_1 \to M_2 \to M_3 \to 0$, the induced sequence $0 \to B \otimes_A M_1 \to B \otimes_A M_2 \to B \otimes_A M_3 \to 0$, is exact.

Definition 4.0.97. The **function field of V** (or the field of rational functions of V) is the quotient field $k(V)$ of $k[V]$.

Definition 4.0.98. Let V be an irreducible quasi-projective variety. Define the **function field of** V, denoted $k(V)$, to be the function field of any (dense) open affine subset of V.

Definition 4.0.99. The **genus** of X, denoted by $g(x)$, is the dimension over k of the space of global differential forms.

Definition 4.0.100. If X is a smooth projective curve the **genus** of X is $dim_x(\Omega_x(X))$.

Definition 4.0.101. Let V, W be irreducible. The **graph of the rational map** $V \overset{\varphi}{\dashrightarrow} W$ is the closure in $V \times W$ of the graph of the regular map $U \xrightarrow{\varphi|_U} W$, where $U \subseteq V$ is open and dense. In particular, $\Gamma_\varphi = \overline{\{(X, \varphi(x)) \mid x \in U\}} \subseteq V \times W$.

Definition 4.0.102. The **homogeneous coordinate ring** $V \subseteq \mathbb{P}^n, k[V] = k[x_0, \ldots, x_n]/\mathbb{I}(V)$, where $\mathbb{I}(V)$ is the ideal generated by homogeneous polynomials vanishing on V.

Definition 4.0.103. If $f \in k[x_1, \ldots, x_n]$, write $f = f_d + f_{d+1} + \cdots + f_{d+t}$ where the degree of f_i is i and $f_i \neq 0$. The **homogenization of degree** $d+t$ of f is $\tilde{f} \in k[x_0, \ldots, x_n]$, where $\tilde{f} = x_0^t f_d + \cdots + x_0 f_{d+t-1} + f_{d+t}$.

Definition 4.0.104. A **hypersurface of degree** d in $\mathbb{P}^n$ is the zeroset of a single degree d, homogeneous polynomial in $(n+1)$-variables.

Definition 4.0.105. Let $p = 0 \in V \subseteq \mathbb{A}^n$ with $\mathbb{I}(V) = (F_1, \ldots, F_m)$ and a line L such that $p \in L = \{t(a_1, \ldots, a_n) = t\underline{a} \mid \underline{a} \neq 0\}$. The **intersection multiplicity of** $L \cap V$ **at** p is the highest power of t dividing the $gcd(F_1(t\underline{a}), \ldots, F_m(t\underline{a})) \in k[t]$.

Definition 4.0.106. A topological space V is **irreducible** if whenever $V = V_1 \cup V_2$, where $V_1, V_2 \subset V$ closed, then $V = V_1$ or $V = V_2$.

Definition 4.0.107. An **isomorphism of affine algebraic sets** are morphism $V \underset{g}{\overset{f}{\rightleftarrows}} W$ such that their composition is the identity, $f \circ g = 1, g \circ f = 1$.

Definition 4.0.108. An **isomorphism of quasi-projective varieties** is a morphism $\varphi : X \to Y$ that admits a (regular morphism) inverse $\psi : Y \to X$ such that $\varphi \circ \psi : Y \to Y = 1_Y$ and $\psi \circ \phi = 1_X$.

Definition 4.0.109. The **Krull dimension** of a commutative ring R is the length of the longest chain of prime ideals of R.

Definition 4.0.110. A **line bundle** on X is a surjective regular map $L \overset{\pi}{\twoheadrightarrow} X$ with the property that X has an open cover U_λ such that:
$\pi^{-1}(U_\lambda) \underset{\cong}{\xrightarrow{\varphi_\lambda}} U_\lambda \times k$ and $\pi^{-1}(U_\mu) \underset{\cong}{\xrightarrow{\varphi_\mu}} U_\mu \times k$, then

$$\begin{array}{ccc} \pi^{-1}(U_\lambda \cap U_\mu) & \underset{\cong}{\xrightarrow{\varphi_\lambda}} & (U_\lambda \cap I_\mu) \times k \\ & \searrow^{\varphi_\mu} & \downarrow{\scriptstyle \varphi_\mu \cdot \varphi_\lambda^{-1}} \\ & & (U_\lambda \cap U_\mu) \times k \end{array}$$

where $\varphi_\mu \cdot \varphi_\lambda^{-1} : (x, \lambda) \mapsto (x, g(x)\lambda)$ for $g \in \mathcal{O}_X^*(U_\lambda \cap U_\mu)$

Definition 4.0.111. A **linear system** on X is a collection of effective linearly equivalent divisors corresponding to some vector subspace of $\mathcal{L}(D)$.

Definition 4.0.112. Divisors D, D' on any X are **linearly equivalent**, denoted $D \sim D'$, if they represent the same class in $Cl(X)$. Equivalently, if $\exists f \in k(X)^*$ such that $D = D' + div f$.

Definition 4.0.113. $f_1, \ldots, f_t \in \mathcal{O}_{X,p}$ are **local defining equations** for closed $Y \subsetneq X$ if $\exists U \subseteq X$ affine open neighborhood of p where f_i are regular and $\mathbb{I}(Y \cap U) \subseteq \mathcal{O}_X(U)$ is generated by the f_i's. Equivalently, if $\{g \in \mathcal{O}_{X,p} \mid g$ vanishes in Y on an open neighborhood of $p\} = \mathbb{I}(V)_p = \mathcal{O}_{X,p}$ is generated by $f_1, \ldots, f_t$.

Definition 4.0.114. Let $Z \subseteq X$ be any irreducible closed subvariety of X. The **local ring of X along Z** is:

$$\begin{aligned}
\mathcal{O}_{X,Z} &= \{\varphi \in k(X) \mid \varphi \text{ is regular at some point of } Z\} \\
&= \{\varphi \in k(X) \mid \text{the domain of definition of}(\varphi) \cap Z \neq \emptyset\} \\
&= \{\varphi \in k(X) \mid \varphi \text{ is regular on a non-empty open set of } Z\} \\
&= \varinjlim_{U \cap Z \neq \emptyset} \mathcal{O}_X(U) \\
&= \mathcal{O}_X(U)[f^{-1} \mid f \notin \mathbb{I}(Z \cap U)] \\
&= \mathcal{O}_X(U)_{\mathbb{I}(Z \cap U)}
\end{aligned}$$

We are allowing rational functions whose denominator does not vanish completely along Z. In the last three equalities we passed to any open affine $U \subseteq X, U \cap Z \neq \emptyset$.

Definition 4.0.115. A subset, V, of a topological space is **locally closed** in X if $V = U \cap C$ where $U \subset X$ is open and $C \subset X$ is closed.

Definition 4.0.116. Given a rational map $V \overset{F}{\dashrightarrow} W$, the **locus of indeterminacy** of F is the set of points at which F is not regular (i.e. undefined).

Definition 4.0.117. A **morphism** (or **regular map**) between affine algebraic sets $V \xrightarrow{f} W$ is simply a mapping (that is the restriction of) a polynomial map on the ambient affine spaces.

Definition 4.0.118. A **morphism (regular map)** $\varphi : X \subset \mathbb{P}^N \to Y \subset \mathbb{P}^M$ of quasi-projective varieties is a map of sets which is locally given by regular functions on affine charts. More precisely, $\forall x \in X, \exists$open $U \subseteq X$ containing x such that $\varphi(U) \subseteq U_i$ and on U, $\varphi : p \in U \to (\varphi_1(p), \ldots, \varphi_M(p)) \in Y \cap U_i \subseteq U_i = \mathbb{A}_M$, where $\varphi_i \in \mathcal{O}_X(U)$ are regular functions on U.

Definition 4.0.119. If $\mathcal{F}$ and $\mathcal{G}$ are presheaves on X, a **morphism** of abelian groups $\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ for each open set U, such that whenever $V \subseteq U$ is an inclusion, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\
\rho_{UV} \downarrow & & \downarrow \rho'_{UV} \\
\mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V)
\end{array}$$

Definition 4.0.120. A ring is **Noetherian** if every ideal is finitely generated

Definition 4.0.121. A domain R is **normal** if it is integrally closed in its fraction field.

Definition 4.0.122. An irreducible variety X is **normal** if $\exists\{U_\lambda\}$ open affine cover of X such that $\mathcal{O}_X(U_\lambda)$ is normal. Equivalently, a variety X is **normal** if $\mathcal{O}_{X,x}$ is normal $\forall x \in X$.

Definition 4.0.123. Let $D = \Sigma_{i=1}^{t} n_i Y_i$ where the Y_i are prime. Then $D \in Div(X)$ is **locally principal** if the ideal $\mathbb{I}_{Y_i} \subseteq \mathcal{O}_X$ is locally principal. In particular, if there exists an open affine cover $\{U_\lambda\}$ of X such that $\mathbb{I}(Y_i \cap U_\lambda) \subseteq \mathcal{X}(U_\lambda)$ is principal.

Definition 4.0.124. Let $U \subset X$ be an open affine set and let $g \in \mathcal{O}_X(U)$. Let $Y \subseteq \mathbb{V}(g)$ be irreducible. The **order of vanishing** of g along Y is $t \in \mathbb{N}$ such that $g \in (\pi_i)^t - (\pi_i)^{t+1}$. Notation: $t = \nu_{Y_i}(g) = ord_{Y_i}(g)$.

Definition 4.0.125. Let p be a smooth point on a variety X of dimension n (at p). Let $u_1, \ldots, u_n$ be regular functions at p that vanish at p. (Note: $u_i \in \mathcal{O}_{X,p}$). Then $u_1, \ldots, u_n$ are **parameters at** p if their images in m/m^2 are a basis for this cotangent space, where m is the maximal ideal of $\mathcal{O}_{X,p}$.

Definition 4.0.126. The **Picard group** of X is the quotient $\frac{CDiv(X)}{P(X)} \equiv Pic(X)$.

Definition 4.0.127. Let X be a topological space. A **presheaf**, $\mathcal{F}$, of abelian groups on X consists of data:

- For every open subset $U \subseteq X$, an abelian group $\mathcal{F}(U)$, and
- For every inclusion $V \subseteq U$ of open subsets of X, a morphism of abelian groups $\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$

subject to the conditions:

(1) $\mathcal{F}(\emptyset) = 0$
(2) ρ_{UU} is the identity map $\mathcal{F}(U) \longrightarrow \mathcal{F}(U)$, and
(3) If $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

Definition 4.0.128. A **presheaf of rings** (or groups, etc), $\mathcal{F}$, on a topological space X is a contravariant functor from the category of open sets of X to the category of rings (or groups, etc.).

Definition 4.0.129. An ideal $P \subseteq R$ is **prime** if $xy \in P \Rightarrow x \in P$ or $y \in P$. Equivalently, P is prime $\Leftrightarrow R/P$ is a domain.

Definition 4.0.130. A **prime divisor** of X is an irreducible codimension 1 (closed) subvariety.

Definition 4.0.131. A **projective algebraic set** (projective variety) $V \subseteq \mathbb{P}_k^n$ is the common zero set of an arbitrary collection $\{F_\lambda\}_{\lambda \in \Lambda}$ of homogeneous polynomials in $k[x_0, \ldots, x_n], V = \mathbb{V}(\{F_\lambda\}_{\lambda\in\Lambda}) \subseteq \mathbb{P}_k^n$.

Definition 4.0.132. If $V \subseteq \mathbb{A}^n$ is an affine algebraic set in $\mathbb{A}^n$, then its **projective closure** $\overline{V}$ is its Zariski-closure, in $\mathbb{P}^n$, under the embedding: $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$ by $(x_1, \ldots, x_n) \mapsto [1 : x_1 : \cdots : x_n]$.

Definition 4.0.133. A **projective morphism** $X \longrightarrow V$ is one that factors as $X \overset{\text{closed}}{\hookrightarrow} V \times \mathbb{P}^n \xrightarrow{\pi_1} V$

Definition 4.0.134. For any vector space V over k, the **projective space of** V, denoted $\mathbb{P}(V)$, is the set of all 1-dimensional subspaces $\mathbb{P}_k^n = \mathbb{P}(k^{n+1})$.

Definition 4.0.135. Let $X \xrightarrow{\varphi} Y$ be a dominant morphism of irreducible varieties. If $D \in CDiv(Y)$ is given by the data $\{U_\lambda, f_\lambda\}$, then the **pull-back of** D, denoted $\varphi^* D$, is given by data $\{\varphi^{-1}(U_\lambda), \varphi^*(f_\lambda)\}$.

Definition 4.0.136. A **quasi-projective variety** is a locally closed subset of $\mathbb{P}_k^n$.

Definition 4.0.137. The **radical** of I is defined as: $RadI \equiv \{f \in R \mid f^n \in I \text{ for some } n\}$

Definition 4.0.138. A **rational differential p-form** is an equivalence class (U, ω) where $U \subseteq X$ is a non-empty open set and ω is a regular p-form on U. Then $(U, \omega) \sim (U', \omega') \Leftrightarrow \omega|_{U \cap U'} = \omega'|_{U \cap U'}$.

Definition 4.0.139. A rational function φ is **regular at** $p \in V$ if $\exists f, g \in k[V]$ such that $g(p) \neq 0$ and $\varphi = \frac{f}{g}$.

Definition 4.0.140. A **rational map** $V \overset{\varphi}{\dashrightarrow} W$ is a regular map on some (unspecified) dense open subset U of V such that $U \xrightarrow{\varphi} W$.

Definition 4.0.141. A **rational map of quasi-projective varieties** $V \overset{F}{\dashrightarrow} W$ is an equivalence class of regular maps $\{U \xrightarrow{\varphi} W \mid U \subseteq V \text{ dense open}\}$ where the equivalence relation is defined by $\{U \xrightarrow{\varphi} W\} \sim \{U' \xrightarrow{\varphi'} W\}$ if $\varphi|_{U \cap U'} = \varphi'|_{U \cap U'}$.

Definition 4.0.142. A **regular function** $g : V \to k$ is a function that agrees with the restriction of a polynomial.

Definition 4.0.143. A **regular map** (or **morphism**) between affine algebraic sets $V \xrightarrow{f} W$ is simply a mapping (that is the restriction of) a polynomial map on the ambient affine spaces.

Definition 4.0.144. Let $W \subseteq \mathbb{P}^n$ be a quasi-projective variety. A function $\varphi : W \to k$ is **regular** on W if for all points $p \in W$, there exists $F_p, G_p \in k[x_0, \ldots, x_n]$ homogeneous of the same degree such that φ agrees with the function $\frac{F_p}{G_p}$ on some neighborhood of p.

Definition 4.0.145. A **regular differential form** on a variety X over k is a differential form w with the property that there exists an open affine cover $\{U_\lambda\}$ of X such that $w|_{U_\lambda}$ agrees with the differential form $g_1 df_1 + \ldots g_t df_t$ for some $g_i, f_j \in \mathcal{O}_X(U_\lambda)$.

Definition 4.0.146. A **regular differential p-form** is an assigment ω which, locally on affine charts, looks like $\Sigma g_{i_1,\ldots,i_p} df_{i_1} \wedge df_{i_2} \wedge \cdots \wedge df_{i_p}$ where $f_i, g_{i_1,\ldots,i_p}$ are regular functions.

Definition 4.0.147. Fix a divisor D on an irreducible, normal variety X. The **Riemann-Roch space** is $\mathcal{L}(D) = \{f \in k(X)^* \mid divf + D \geq 0\} \cup \{0\}$.

Definition 4.0.148. Fix $V \subseteq \mathbb{A}^n$ irreducible affine algebraic set of over $k = \overline{k}$ and take an open set $U \subseteq V$. The **ring of regular functions on** U is the ring of all rationals functions that are regular on U. This is denoted by $\mathcal{O}_V(U) = \{\varphi \in k(V) \mid \varphi \text{ is regular at each } p \in U\}$.

Definition 4.0.149. A projective variety, $V \subseteq \mathbb{P}^n$, is a **scheme theoretic complete intersection** if $\mathbb{I}(V) \subseteq k[x_0, \ldots, x_n]$ is generated by $N - dimV$ polynomials.

Definition 4.0.150. A **section** of a line bundle $L \xrightarrow{\pi} X$ over $U \subseteq X$ is a (regular) map $U \xrightarrow{s} L$ such that $\pi \circ s = id|_U$. These form a sheaf $\mathcal{L}(U) = $ sections on U.

Definition 4.0.151. A projective variety $V \subseteq \mathbb{P}^n$ is a **(set-theoretic) complete intersection** if it has codimension r and it is defined by r homogeneous equations.

Definition 4.0.152. A presheaf $\mathcal{F}$ on a topological space X is a **sheaf** if it satisfies the following conditions:

(1) If U is an open set, $\{V_i\}$ is an open covering of U, and if $s \in \mathcal{F}(U)$ is an element such that $s|_{V_i} = 0 \forall i$, then $s = 0$. (Note that this condition implies that s is unique)
(2) If U is an open set, $\{V_i\}$ is an open covering of U, and if we have elements $s_i \in \mathcal{F}(V_i) \forall i$, with the property that for each i, j we have $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i \forall i$.

Definition 4.0.153. A point p on a quasi-projective variety V is a **smooth** point of V if $dimT_pV = dim_pV$. Otherwise, p is a **singular point.**

Definition 4.0.154. If $\mathcal{F}$ is a presheaf on X, and if P is a point of X, we define the **stalk** $\mathcal{F}_p$ of $\mathcal{F}$ at P to be the direct limit of the groups $\mathcal{F}(U)$ for all open sets U containing P via the restriction maps ρ.

Definition 4.0.155. Let $D = \Sigma_{i=1}^{t} n_i Y_i$ be a divisor where $n_i \neq 0$ and Y_i is a pure codimension one subvariety of X. The **support of** D, $SuppD$, is $Y_1 \cup \cdots \cup Y_t$.

Definition 4.0.156. The line L is **tangent to** V **at** p if the intersection multiplicity of L and V at p, $(L \cdot V)_p$, is greater than or equal to two.

Definition 4.0.157. The **tangent space to** V **at** p, denoted T_pV, is the set of all points lie on a line tangent to V at p.
In particular, $T_pV = \{(x_1, \ldots, x_n) \in \mathbb{A}^n \mid (x_1, \ldots, x_n) \in L$, where L is a line tangent to V at $p\}$

Definition 4.0.158. A maximal set of algebraically independent elements of L/k is a **transcendence basis for** L/k. The cardinality of any 2 transcendence basis is the same, it is called the transcendence degree.

Definition 4.0.159. A **very ample** linear system $|W|$ is one where $\varphi_{|W|}$ is an embedding.

Definition 4.0.160. The **Zariski tangent space** to a point p on a quasi-projective variety V will be defined $(m_p/m_p^2)^*$, where $m_p \subseteq \mathcal{O}_{V,p}$ is the maximal ideal of regular functions vanishing at p.