Algebraic Geometry II

Notes by: Sara Lapan
Based on lectures given by: Karen Smith

Contents

[1]Books recommended for this course included, but were not limited to:
(1). *Basic Algebraic Geometry 2: Schemes and Complex Manifolds* by Igor R. Shafarevich and M. Reid
(2). *Algebraic Geometry* by Robin Hartshorne
(3). *The Geometry of Schemes* by David Eisenbud and Joe Harris

[2]These notes were typed during lecture and edited somewhat, so be aware that they are not error free. if you notice typos, feel free to email corrections to swlapan@umich.edu.

Part 1. Introduction to Schemes

1. Modernizing Classical Algebraic Geometry

Lecture 1. January 8, 2009

Read Shaf. II: V 1.1-1.3, Exercises p15: 1,3,4,5 (due Tuesday)
Books for the course: Hartshorne, Shaf. II, and Geometry of Schemes

Classical Algebraic Geometry:
The main object of study is an algebraic variety over a fixed algebraically closed field.
An algebraic variety, X:

- is a topological space with a cover of open sets that are affine algebraic varieties
- comes with a sheaf of rings: on each open set U, $\mathcal{O}_X(U)$ =the ring of regular functions on U

A scheme, X:

- is a topological space with a cover of open sets, each of which is an affine scheme
- comes with a sheaf of rings $\mathcal{O}_X$

An affine scheme is of the form (Spec $A, \widetilde{A}$), where $\widetilde{A}$ is the same ring as A but it is viewed as all functions on Spec A. A scheme need not be defined over anything, whereas an algebraic variety is defined over a fixed algebraically closed field. However, one often assumes that the scheme is over a fixed algebraically closed field. A scheme can also be defined over a ring, such as $\mathbb{Z}$.

Local Picture of Classical Algebraic Geometry
Fix $k = \overline{k}$. $X = \mathbb{V}(\{F_\lambda\}_{\lambda\in\Lambda}) \subseteq k^n$ is an affine variety (without loss of generality) we can assume that Λ is finite and that $\{F_\lambda\}_{\lambda\in\Lambda}$ generates a radical ideal.

Definition 1.1. A **morphism** between affine algebraic varieties $X \xrightarrow{\varphi} Y$ is a map that agrees with the restriction of some polynomial map on the ambient spaces at each point.

Definition 1.2. Given an affine algebraic variety $X \subseteq k^n$, the **coordinate ring** of X, denoted $\mathcal{O}_X X$, is the ring of regular functions on X, which in this case is simpy functions $X \xrightarrow{\varphi} k$ that agree with the restriction of some polynomial on k^n.

Theorem 1.3 (Hilbert's Nullstellensatz or The Fundamental Theorem of Elementary Classical Algebraic Geometry)**.** *The assignment $X \rightsquigarrow \mathcal{O}_X(X)$ defines an anti-equivalence (i.e. a contravariant functor) of categories:*

$$\begin{array}{rcr} \{\textit{Affine varieties over } k\} \longrightarrow & & \{\textit{Reduced, finitely-generated } k\textit{-algebras}\} \\ \mathbb{V}(I) \subset k^n \longrightarrow & & A = k[x_1, \dots, x_n]/I \\ X \longleftarrow & & k[X]/\mathbb{I}(X) \end{array}$$

Since $k[x_1, \dots, x_n] \twoheadrightarrow \mathcal{O}_X(X)$ has kernel $\mathbb{I}(X) = \{g \in k[x_1, \dots, x_n] \mid g|_X = 0\}$. The functor is just the pull-back map:

$$\{X \xrightarrow{\varphi} Y\} \Longleftrightarrow \{\mathcal{O}_X(X) \xleftarrow{\varphi^*} \mathcal{O}_Y(Y), g\circ\varphi \longleftarrow g\}$$

The category on the right looks rather specific, so in scheme theory we want to remove some of those "arbitrary" restrictions.

Remark 1.4. In this class, a ring is always a commutative ring with identity.

$$
\begin{array}{ccc}
\{\text{Affine varieties over } k\} & \longleftrightarrow & \{\text{Reduced, finitely-generated } k\text{-algebras}\} \\
\downarrow \subseteq & & \downarrow \subseteq \\
\{\text{Affine schemes over } k\} & \longleftrightarrow & \{k-\text{algebras}\} \\
\downarrow \subseteq & & \downarrow \subseteq \\
\{\text{Affine Schemes}\} & \longleftrightarrow & \{\text{Rings}\}
\end{array}
$$

Definition 1.5. Fix A, a commutative ring with identity. As a set, **Spec** A is the set of prime ideas of A. We consider it as a topological space with the Zariski topology. The closed sets are: $\mathbb{V}(\{F_\lambda\}_{\lambda\in\Lambda}) = \{p \in \text{Spec } A \mid p \supseteq \{F_\lambda\}_{\lambda\in\Lambda}\}$. Without loss of generality we can replace $\{F_\lambda\}_{\lambda\in\Lambda}\}$ by any collection of elements generating the same ideal or even generating any ideal with the same radical.

Example 1.6 (Spec $\mathbb{Z}$). The closed points are the maximal ideals, so they are (p), where p is a prime number. The closure of the point (0) is Spec $\mathbb{Z}$. Since $\mathbb{Z}$ is a PID, all of the closed sets will be of the form $\mathbb{V}(n) = \{(p_1), \dots, (p_r)\}$, where $n = p_1^{a_1} \dots p_r^{a_r}$. Note that the topology on Spec $\mathbb{Z}$ is not the finite complement topology since it contains a dense point, so we look at the maximal ideals for the open topology.

Example 1.7. Let $A = \frac{k[x_1,\dots,x_n]}{(F_1,\dots,F_r)}$ be reduced, where $k = \overline{k}$. The maximal ideals of A are $(x_1 - a_1, \dots, x_n - a_n)$, where $(a_1, \dots, a_n) \in \mathbb{V}(F_1, \dots, F_r) \subseteq k^n$. Since closed points of Spec A are in one-to-one correspondence with maximal ideals of A, the closed points of Spec A are the points of the affine variety $\mathbb{V}(F_1, \dots, F_r)$. Due to the bijection between prime ideals in A and irreducible subvarieties, the points of Spec A correspond to irreducible (closed) subvarieties of $\mathbb{V}(F_1, \dots, F_r)$.

Proposition 1.8. *If $N \subseteq A$ is the ideal of nilpotent elements, then Spec $A \cong_{homeo}$ Spec A/N as a topological space (but different scheme unless $N = 0$).*

Example 1.9. Spec $\frac{k[x]}{(x^2)}$ is a topological space with one point $(\overline{x})$.

Example 1.10. Now let's consider Spec $(k \oplus kx)$. Since $k \oplus kx \cong^{\text{v.s}} \frac{k[x]}{(x^2)}$, this is the same topological space as the previous example but not the same scheme.

Let $R = \frac{k[x]}{(x^2)}$ and $S = \frac{k[x]}{(x)}$. The map $R \xrightarrow{\varphi} S$ is given by killing x. Since $\frac{k[x]}{(x^2)} \cong k \oplus kx$ and $\frac{k[x]}{(x} \cong k$, $k[x] \longrightarrow \frac{k[x]}{(x^2)}$ is given by restriction:

$$f = a_0 + a_1 x + a_2 x^2 + \dots \mapsto f \mod x^2 = a_0 + a_1\overline{x}, \text{ where } a_0 = f(0), a_1 = \frac{\partial f}{\partial x}(0)$$

Spec $\frac{k[x]}{(x^2)}$ intuitively is the origin in $\mathbb{A}^1$ together with a "first-order neighborhood."

Lecture 2. January 13, 2009

Exercise 1.11. Due Tuesday: Shaf. §1: 6,7,8 and §2:1. Read up on sheaves in Hartshorne
Main Starting Point for Scheme Theory:

$$\{\text{Rings}\} \xrightarrow{\text{contravariant functor}} \{\text{Topological Spaces}\}$$

Proposition 1.12. *If $A \xrightarrow{\varphi} B$ is a ring homomorphism, then Spec $B \xrightarrow{{}^a\varphi}$ Spec A sending $p \mapsto \varphi^{-1}(p)$ is a continuous map of topological spaces.*

Proof. We need the preimage of a closed set $\mathbb{V}(I) \in \text{Spec } A$ to be closed.

$$p \in ({}^a\varphi)^{-1}(\mathbb{V}(I)) \Leftrightarrow {}^a\varphi(p) \in \mathbb{V}(I) \tag{1}$$
$$\Leftrightarrow \varphi^{-1}(p) \in \mathbb{V}(I) \tag{2}$$
$$\Leftrightarrow \varphi^{-1}(p) \supseteq I \tag{3}$$
$$\Leftrightarrow p \supseteq \varphi(I) \tag{4}$$
$$\Leftrightarrow p \in \mathbb{V}(\varphi(I)) = \mathbb{V}(\varphi(I)B) \tag{5}$$

Where (2) $\Leftrightarrow$ (3) since $\mathbb{V}(I) = \{p \in \text{Spec } A \mid I \subseteq p\}$. □

Corollary 1.13. $({}^a\varphi)^{-1}(\mathbb{V}(I)) = \mathbb{V}(\varphi(I)B) =$*the ideal generated by* $\varphi(I) \subseteq B$.

Crucial Example 1:
The quotient homomorphism $A \twoheadrightarrow A/I$ induces a homeomorphism $\text{Spec } A/I \xrightarrow{\cong} \mathbb{V}(I) \subseteq \text{Spec } A$. This is a quintessential example of a closed embedding of schemes.
Crucial Example 2:
The homomorphism $A \longrightarrow A[\frac{1}{f}] = A$ localized at the multiplicative system $\{1, f, f^2, \ldots\}$ induces a homeomorphism $\text{Spec } A[\frac{1}{f}] \longrightarrow D(f) = \{p \in \text{Spec } A \mid f \notin p\} = \text{Spec } A - \mathbb{V}(f)$. This is a quintessential example of an open embedding of schemes.

Proposition 1.14. *The open sets* $\{D(f)\}_{f \in A}$ *form a basis for the Zariski-topology on Spec A.*

Proof. Take an arbitrary open set U. For some ideal $I \subseteq A$:

$$\begin{aligned} U &= \text{Spec } A - \mathbb{V}(I) \\ &= \{p \mid p \not\supseteq I\} \\ &= \{p \mid \exists f \in I, f \notin p\} \\ &= \cup_{f \in I} D(f) \end{aligned}$$

□

Facts/Exercises:

(1) If A is Noetherian, every open set of Spec A is a compact topological space. Even if A is not Noetherian, Spec A is compact.
(2) Spec A is irreducible as a topological space $\Leftrightarrow A/Nil(A)$ is a domain $\Leftrightarrow Nil(A) = \{f | f^n = 0\}$ is prime
(3) Spec A is disconnected $\Leftrightarrow A \cong A_1 \times A_2$, where $A_1, A_2 \neq \varnothing$

Next we want to define the structure sheaf of an affine scheme Spec A. Given $X = \text{Spec } A$, we want a sheaf of rings, $\mathcal{O}_X$, on X. In particular, for any open set $U \subseteq X$ we want $\mathcal{O}_X(U)$ to be a ring where for any open set $V \subseteq U$, the map $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$ is a ring homomorphism given by restriction.

Classical Algebraic Geometry:
Let $X \subseteq \mathbb{A}^n$ be an affine algebraic variety over k. Then $k[X]$ is the coordinate ring of X. Let $U \subseteq X$ be an open set. Then $\mathcal{O}_X(U) = \{\varphi : U \longrightarrow k \mid \varphi \text{ is regular at each point of } U\}$ and for an open set $V \subseteq U$, $\mathcal{O}_X(V) \longleftarrow \mathcal{O}_X(U)$ by $\varphi|_V \longleftarrow \varphi$.
Main Features of $\mathcal{O}_X$:

- $\mathcal{O}_X(X) = k[X]$
- $D(f) = X - \mathbb{V}(f) \subseteq X \rightsquigarrow \mathcal{O}_X(D(f)) = k[X][\frac{1}{f}]$

- Restriction looks like localization: $D(f) \subseteq X$, then
$$k[X][\frac{1}{f}] = \mathcal{O}_X(D(f)) \longleftarrow \mathcal{O}_X(X) = k[X] \text{ by } \frac{\varphi}{1} \longleftarrow \varphi.$$
- In practice, we only need to consider $\mathcal{O}_X(U)$ for U in a basis of X.

Modern Algebraic Geometry:
Let $X = \text{Spec } A$. We want $\mathcal{O}_X$ to have the same features as in classical algebraic geometry:

- $\mathcal{O}_X(X) = A$
- $\mathcal{O}_X(D(f)) = A[\frac{1}{f}]$.
- The inclusion $D(f) \subseteq X$ induces
$$\mathcal{O}_X(X) \xrightarrow{\text{restriction}} \mathcal{O}_X(D(f))$$
$$A \xrightarrow{\text{localization}} A[\frac{1}{f}]$$
$$a \longrightarrow \frac{a}{1}$$
- Let A be a finitely generated reduced k-algebra, where $k = \overline{k}$. For an open set $U \subseteq \text{Spec } A$, $\mathcal{O}_X(U)$ is the ring of regular functions on $\widetilde{U} = U \cap \text{maxSpec}A$ from classical geometry. Note that $\widetilde{U} \subseteq \text{maxSpec}A$ because in classical algebraic geometry we only consider maximal ideals.

If $X \subseteq \mathbb{A}^n$ is an irreducible affine variety over k, then $k[X]$ is a domain and
$$\mathcal{O}_X(U) = \{\varphi : U \to k | \forall p \in U \exists f, g \in k[X] \text{ with } g(p) \neq 0 \text{ and } \varphi = \frac{f}{g} \text{ in a neighborhood of } p\}$$

Constructing $\mathcal{O}_X$:
First Case: Let $X = \text{Spec } A$, where A is a domain with fraction field L, and $U \subset X$ open. Consider:
$$\mathcal{O}_X(U) = \{\varphi | \forall p \in U \exists f, g \in A \text{ such that } \varphi \text{ has a representative } \frac{f}{g} \text{ with } g(p) \neq 0\} \subseteq L$$
What does it mean for $g(p) \neq 0$?
We need to interpret elements of A as functions on Spec A.
In the classical case:
For $g \in k[X]$ and $p = (\lambda_1, \ldots, \lambda_n) \in X$, $g(p)$ was the image of g under the natural map $k[X] \longrightarrow k$ given by evaluation at p. This map has kernel the maximal ideal $m_p = \{f \in k[X] \mid f(p) = 0\}$. So $g(p) = g \mod m_p \in \frac{k[X]}{m_p} \cong k$.

Definition 1.15. The value of $a \in A$ at $p \in \text{Spec } A$ is $a(p) =$ image of a under the natural map:
$$A \longrightarrow k(p) = \frac{A_p}{pA_p} = \text{ fraction field of } A/p = \text{ residue field of } p$$

Remark 1.16. If m_p is the maximal ideal of a point p on an affine variety $A = k[X]$, then this definition agrees with evaluation at a point. Oddly, the values of a live in different fields as p varies.

Now we can define $\mathcal{O}_X(U)$ more precisely:
$$\begin{aligned}\mathcal{O}_X(U) &= \{\varphi | \forall p \in U \subseteq \text{Spec } A \exists \text{ representative } \varphi = \frac{f}{g} \text{ with } g \notin p\} \\ &= \{\varphi | \forall p \in U \subseteq \text{Spec } A \exists \text{ representative } \varphi = \frac{f}{g} \text{ with } f, g \in A \text{ and } g(p) \neq 0\}\end{aligned}$$

Proposition 1.17. *If A is a domain and $\mathcal{O}_X(U)$ is given as above, then $\mathcal{O}_X(X) = A$.*

Proof. Let $A \longrightarrow \mathcal{O}_X(X)$ by $a \mapsto \frac{a}{1}$. Take any $\varphi \in \mathcal{O}_X(X)$ and at p, write $\varphi = \frac{a_p}{b_p}$ so that $b_p \notin p$. Then the ideal $(\{b_p\}_{p \in \text{Spec } A})$ is not contained in any prime/maximal ideal, hence it must be the unit ideal. Therefore we can write 1 as $1 = r_1 b_{p_1} + \ldots + r_t b_{p_t}$ and so $\varphi = a_{p_1} r_1 + \ldots + a_{p_t} r_t \in A$. Similarly, $\mathcal{O}_X(D(f)) = A[\frac{1}{f}]$. ☐

Lecture 3. January 14, 2009

Remark 1.18. For an arbitrary A, if $a \in A$ satisfies $a(p) = 0 \forall p \in \text{Spec } A$ then $a \in Nil(A)$.

A sheaf needs local information. An example of a pre-sheaf that is not a sheaf is all integrable functions on $\mathbb{R}$, whereas the corresponding sheaf would be all locally integrable functions.

Definition 1.19. A presheaf $\mathcal{F}$ on a topological space X is a **sheaf** if it satisfies the SHEAF AXIOM:
Given $U = \cup_{\lambda \in \Lambda} U_\lambda$, and sections $s_\lambda \in \mathcal{F}(U_\lambda)$ which have the property that for $s_\lambda \in \mathcal{F}(U_\lambda$ and $s_\mu \in \mathcal{F}(U_\mu)$, $s_\lambda|_{U_\lambda \cap U_\mu} = s_\mu|_{U_\lambda \cap U_\mu} \in \mathcal{F}(U_\lambda \cap U_\mu)$, then there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_\lambda} = s_\lambda$ and $s|_{U_\mu} = s_\mu$.

Definition 1.20. Let X be a topological space and $\mathcal{O}_X$ be a sheaf of rings on X. Then $(X, \mathcal{O}_X)$ is a **ringed space**.

Remark 1.21. Let $X = \text{Spec } A$. With the above definition of $\mathcal{O}_X$, $(X, \mathcal{O}_X)$ is a ringed space.

Digression on Limits: Fix a partially ordered index set I. Typically, we assume that I is directed so that $\forall i, j \in I \exists k$ such that $k \geq i, j$.

Definition 1.22. A **direct limit system** (of rings or of objects in any category) is given by a collection of objects $\{A_i\}_{i \in I}$ and morphisms $A_j \xrightarrow{\varphi} A_i$ for $i \geq j$. These morphism satisfy the property that if $i \geq j \geq k$, then we have maps $A_j \longrightarrow A_i$, $A_k \longrightarrow A_j$ and $A_k \longrightarrow A_i$, where the latter is given by composition.

Definition 1.23. The **direct limit** (if it exists) is $\sqcup A_i / \sim$ where for $a_i \in A_i$ and $a_j \in A_j$, $a_i \sim a_j$ if $\exists k \geq i, j$ and a_i, a_j both map to the same element in A_k.

Examples:

- If the A_i are open sets of a topological space and the ordering is given by inclusion, then direct limit: $\varinjlim A_i = \cup A_i$
- If $A_i = k[x_1, \ldots, x_i]$, take $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$, then the direct limit of A_i's is $k[x_1, \ldots, x_i, \ldots]$.
- Let $X \subseteq \mathbb{A}^n$ be an affine algebraic variety over $k = \overline{k}$ and $p \in X$. Then $\mathcal{O}_{X,p} = \varinjlim_{p \in U} \mathcal{O}_X(U)$.
- Let $U \subseteq A$ be a multiplicative system and $S(U)$ be the elements in $k[X]$ that do not vanish on U. Then $\varinjlim_{f \in U} A[\frac{1}{f}] = A[S(U)^{-1}]$

Elements of a direct limit $\varinjlim_i A_i$ are represented by (A_i, a_i) where $(A_i, a_i) \sim (A_j, a_j)$ if a_i and a_j eventually map to the same element in some A_k, $k >> 0$.

Definition 1.24. An **inverse limit system** (or projective or indirect) is a collection of objects $\{A_i\}_{i \in I}$ and morphisms $A_i \longrightarrow A_j$ whenever $i \geq j$.

Definition 1.25. The **inverse limit** (if it exists), $\varprojlim A_i$ is the subobject of $\prod_{i \in I} A_i$ consisting of elements $(a_i)_{i \in I}$ where $a_i \mapsto a_j$ whenever $i \geq j$.

Examples:

- If $\{U_i\}_{i \in I}$ is a collection of open sets in a topological space X and $\{X_i\}_{i \in I} \subseteq \prod U_i$, $U_2 \subseteq U_1, X_2 \mapsto X_1$. So if the inverse limit exists, then $\varprojlim U_i = \cap_{i \in I} U_i$

- Consider $A_1 = k[X]/(X) \twoheadleftarrow A_2 = k[X]/(X^2) \twoheadleftarrow \ldots$, then $\varprojlim A_i = k[[X]]$.

The Universal Property of Projective Limits: The inverse limit, if it exists, is the one object from which everything maps. For instance, suppose C maps (functorally) to all A_i in a limit system, then C maps through $\varprojlim A_i$.

Defining $\mathcal{O}_X$ for an arbitrary ring:

Lemma 1.26. *Let A be a ring. $D(f) \subseteq D(g) \Leftrightarrow f^n = gh$ for some $n > 0, h \in A$.*

Proof.

$$\begin{aligned} D(f) \subseteq D(g) &\Leftrightarrow \mathbb{V}(f) \supseteq \mathbb{V}(g) \\ &\Leftrightarrow (f) \in Rad(g) \\ &\Leftrightarrow f^n \in (g) \\ &\Leftrightarrow f^n = gh \text{ for some } h \end{aligned}$$

□

Corollary 1.27. *Whenever $D(f) \subseteq D(g)$, there's a natural ring map $A[\frac{1}{g}] \longrightarrow A[\frac{1}{f}]$.*

Proof. Universal property of localization: g is invertible in $A[\frac{1}{f}]$ (since $gh = f^n$), so by the universal property we have the map $A[\frac{1}{g}] \longrightarrow A[\frac{1}{f}]$. □

This produces a limit system: $\{D(g)\} \rightsquigarrow \{A[\frac{1}{g}]\}$, which is a map from a direct limit system of open sets to an inverse limit system.

$$\textbf{Define: } \mathcal{O}_X(U) = \varprojlim_{D(g)\subseteq U} A[\frac{1}{g}]$$

Remark 1.28. When A is a domain, $A \subseteq L$ =fraction field, then:

$$\begin{aligned} \mathcal{O}_X(U) &= \{\varphi | \forall p \in U \subseteq \text{Spec } A \exists \text{ representative } \varphi = \frac{f}{g} \text{ with } g \notin p\} \\ &= \cap_{D(g)\subseteq U} \mathcal{O}_X(D(g)) \\ &= \cap_{A[\frac{1}{g}]\subseteq U} \mathcal{O}_X(A[\frac{1}{g}]) \end{aligned}$$

So these different definitions agree with each other.

Using this definition, we should verify:

$$\mathcal{O}_X(D(f)) = A[\frac{1}{f}] \text{ and } \mathcal{O}_X(\text{Spec } A) = \mathcal{O}_X(D(1)) = A$$

If $U = D(f)$ and we consider $\{D(g)\}_{D(g)\subseteq D(f)}$, then $A[\frac{1}{f}]$ maps to all of the objects in the inverse limit system so that $\varprojlim A[\frac{1}{g}] = A[\frac{1}{f}]$.

Summary: For any commutative ring A and $X = \text{Spec } A$, $(X, \mathcal{O}_X)$ is a ringed space.

Definition 1.29. Given a sheaf $\mathcal{F}$ on a topological space X and $p \in X$, the **stalk** of $\mathcal{F}$ at p, denoted by $\mathcal{F}_p$, is $\varinjlim_{U\ni p} \mathcal{F}(U)$.

Let $X = \text{Spec } A$ and take any $p \in X$. Then:

$$\mathcal{O}_{X,p} = \varinjlim_{p\in U} \mathcal{O}_X(U) = \varinjlim_{p\in D(f)} \mathcal{O}_X(D(f)) = \varinjlim_{f\notin p} A[\frac{1}{f}] = A[(A-p)^{-1}] = A_p$$

Definition 1.30. A **locally ringed space** is a ringed space $(X, \mathcal{O}_X)$ in which $\mathcal{O}_{X,p}$ is a local ring $\forall p \in X$.

Lecture 4. January 20, 2009

Exercise 1.31. Hartshorne II: 1.4,1.6,1.14,1.15,1.17,2.7,2.9,2.11. Due Tuesday on 1/27/2009. Read the more thorough explanation of f^{-1}, f_*, and f^* posted on Karen's website.

Example 1.32. Let $X = \text{Spec } \mathbb{Z}$. The closed sets are: $\{p_1, \ldots, p_t\}, \varnothing, X$ while the open sets are: $D(N) = X \setminus \mathbb{V}(N)$.
The open sets containing p are: $\{U \ni p \text{ open }\} = \{D(N) \ni p\} = \{D(N) \mid N \notin p\}$ so that

$$\mathcal{O}_{X,p} = \varinjlim_{U \ni p} \mathcal{O}_X(U) = \varinjlim_{D(N) \ni p} \mathcal{O}_X(D(N)) = \varinjlim_{N \notin (p)} \mathbb{Z}[\frac{1}{N}] = \mathbb{Z}_{(p)}$$

Example 1.33. Let $R = k[[t]]$ and $X = \text{Spec } R$. Every element in R can be written as a unit times some power of t so that the points are $\{(0), (t)\}$. The closed sets are: $(t), X, \varnothing$, the open sets are: $(0), \varnothing, X$, and the dense point is: (0). Note that $(0) = D(t)$ since it is the only prime ideal not containing (t).

$$\mathcal{O}_X(\varnothing) = 0 \qquad \mathcal{O}_X(X) = R \qquad \mathcal{O}_X((0)) = \mathcal{O}_X(D(t)) = R[\frac{1}{t}] = k[[t]][\frac{1}{t}] = k((t))$$

Stalks:

$$\mathcal{O}_{X,(0)} = \varinjlim_{U \ni (0)} \mathcal{O}_X(U) = \mathcal{O}_X((0)) = k((t))$$

$$\mathcal{O}_{X,(t)} = \varinjlim_{U \ni (t)} \mathcal{O}_X(U) = \mathcal{O}_X(X) = k[[t]]$$

Fix a topological space X.

Definition 1.34. A **morphism** of sheaves of rings (or objects in any category) $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ is a collection of maps of rings (or morphisms in the category) given for each open set $U \subseteq X$ (i.e. $\mathcal{F}(U) \xrightarrow{\varphi(U)} \mathcal{G}(U)$) that is compatible with restriction (i.e. whenever we have an inclusion of open sets $V \subseteq U$, we have aa commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V) \end{array}$$

An isomorphism of sheaves is a morphism $\mathcal{F} \longrightarrow \mathcal{G}$ and $\mathcal{F} \longleftarrow \mathcal{G}$ that when composed gives the identity.

Example 1.35. Let $X = \mathbb{R}^n$ with the Euclidean topology. $C^\infty_{\mathbb{R}^n}$ is a sheaf of C^∞-functions (or a sheaf of rings, of $\mathbb{R}$-algebras, of abelian groups, or of $\mathbb{R}$-vector spaces).

$$\begin{array}{ccc} C^\infty_{\mathbb{R}^n} & \xrightarrow{\frac{\partial}{\partial x_1}} & C^\infty_{\mathbb{R}^n} \\ f & \mapsto & \frac{\partial}{\partial x} f \end{array}$$

is a morphism of sheaves of $\mathbb{R}$-vector spaces but not a morphism of sheaves of rings since differentiation does not preserve multiplication.

Example 1.36. Let $X = \mathbb{A}^n_{\mathbb{C}}$ so that X is an algebraic variety in the classical sense.

$$\underline{\mathbb{C}}(U) = \{ \text{ locally constant } \mathbb{C}\text{-valued functions } U \longrightarrow \mathbb{C}\} \quad \longrightarrow \quad \mathcal{O}_X$$

$$\underline{\mathbb{C}}(U) \longrightarrow \mathcal{O}_X(U) \text{ is given by } \lambda \mapsto \lambda$$

is a morphism of sheaves of $\mathbb{C}$-algebras.

Remark 1.37. The constant sheaf refers to the sheaf of locally constant functions.

Proposition 1.38. *A morphism of sheaves of rings $\mathcal{F} \longrightarrow \mathcal{G}$ on a topological space induces a morphism of stalks $\mathcal{F}_p \longrightarrow \mathcal{G}_p \forall p \in X$.*

Proof. $\varprojlim_{p \in U} \mathcal{F}(U) = \mathcal{F}_p$ and $\varprojlim_{p \in U} \mathcal{G}(U) = \mathcal{G}_p$

$$\begin{array}{ccc}
\mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\
\downarrow & & \downarrow \\
\mathcal{F}(U') & \longrightarrow & (U') \\
\vdots & & \vdots \\
\mathcal{F}_p & \xrightarrow{\varphi_p} & \mathcal{G}_p
\end{array}$$

□

Remark 1.39. If your category (i.e. rings) admits direct limits, then stalks are in that category and the morphism of stalks is a morphism in that category.

Definition 1.40. If $X \xrightarrow{f} Y$ is a continuous map of topological spaces and $\mathcal{F}$ is a sheaf (i.e. of rings) on X, the **direct image sheaf** $f_*\mathcal{F}$ is a sheaf (i.e. of rings) on Y defined as: Let $U \subseteq Y$ be open and $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$. If $V \subseteq U$ is open, then $f^{-1}(V) \subseteq f^{-1}(U)$ and so with the obvious restrictions:

$$\begin{array}{cc}
f_*\mathcal{F}(U) & = \mathcal{F}(f^{-1}(U)) \\
\downarrow & \downarrow \\
f_*\mathcal{F}(V) & = \mathcal{F}(f^{-1}(V))
\end{array}$$

Example 1.41 ("Skyscraper Sheaf"). Let X be a topological space, $p \in X$, G a sheaf of abelian groups on $\{p\}$ and $\{p\} \xrightarrow{f} X$.

$$f_*G(U) = \begin{cases} G(f^{-1}(U)) = 0 & \text{if } p \notin U \\ G & \text{if } p \in U \end{cases}$$

Stalks:

$$(f_*G)_x = \varinjlim_{U \ni x} f_*G(U) = \varinjlim_{U \ni x} G(f^{-1}(U)) = \begin{cases} G & \text{if } x = p \\ 0 & \text{if } x \notin \overline{p} \\ G & \text{if } x \in \overline{p} \end{cases}$$

This is related to exercise 1.17 in Hartshorne.

Definition 1.42. A **morphism of ringed spaces** $(X, \mathcal{O}_X) \xrightarrow{(f,f^\#)} (Y, \mathcal{O}_Y)$ is a continuous map of topological spaces $X \xrightarrow{f} Y$ together with a morphism of sheaves of rings $\mathcal{O}_Y \xrightarrow{f^\#} f_*\mathcal{O}_X$.

Remark 1.43. In practice, $f^\#$ is usually a pullback, but it need not be a pullback.

Example 1.44. Let $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ by $(x,s) \mapsto x$ (where $\mathbb{R}^2$ has the Euclidean topology). We are looking at the ringed spaces: $(\mathbb{R}^2, C^0_{\mathbb{R}^2})$ and $(\mathbb{R}, C^0_{\mathbb{R}})$. There is a map of sheaves: $C^0_{\mathbb{R}}(U) \longrightarrow f_*C^0_{\mathbb{R}^2}(U) = C^0_{\mathbb{R}^2}(U \times \mathbb{R})$ given by $g \mapsto g \circ f$. The morphism of ringed spaces is: $(\mathbb{R}^2, C^0_{\mathbb{R}^2}) \xrightarrow{(f,f^\#)} (\mathbb{R}, C^0_{\mathbb{R}})$, where $f^\#$ is the "pull-back" of f.

Example 1.45. Let $X \xrightarrow{f} Y$ be a morphism of algebraic varieties over an algebraically closed field k This always induces a morphim of ringed spaces:

$$(X, \mathcal{O}_X) \xrightarrow{(f,f^\#)} (Y, \mathcal{O}_Y)$$

via a pullback of regular functions:

$$\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$$
$$\mathcal{O}_Y(U) \longrightarrow f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$$
$$g \mapsto f^*g = g \circ f$$

Check that restriction commutes with pullback. One can check that this is a morphism of ringed spaces and a morphism of locally ringed spaces.

Definition 1.46. A **morphism of locally ringed spaces** $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces with the property that whenever $p \in X$ with $f(p) = q \in Y$, the induced map of stalks $\mathcal{O}_{Y,q} \xrightarrow{f_p} \mathcal{O}_{X,p}$ is a local homomorphism of local rings (so that $f_p(m_{\mathcal{O}_{Y,q}}) \subseteq m_{\mathcal{O}_{X,p}}$).

Remark 1.47. It is possible to have multiple points in X that map to the same point in Y, in which case there will be multiple induced maps of stalks.

Remark 1.48. If $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces and for $p \in X$, $f(p) = q \in Y$, then there is an induced map of stalks:

$$\mathcal{O}_Y \xrightarrow{f^\#} f_*\mathcal{O}_X$$
$$\mathcal{O}_{Y,q} \longrightarrow (f_*\mathcal{O}_X)$$
$$\mathcal{O}_{Y,q} = \varinjlim_{q \in U} \mathcal{O}_Y(U) \xrightarrow{f^\#} \varinjlim_{q \in U} \mathcal{O}_X(f^{-1}(U)) \longrightarrow \varinjlim_{p \in V} \mathcal{O}_X(V) = \mathcal{O}_{X,p}$$

where U and V are open sets.

2. Introducing Schemes

Definition 2.1. An **affine scheme** is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic (as a locally ringed space) to $(\operatorname{Spec} A, \widetilde{A})$ for some commutative ring A.

Lecture 5. January 21, 2009

We want to generalize abstract (non-embedded) algebraic varieties over an algebraically closed field to schemes, which allow nilpotents.
A Brief timeline: Riemann (in 1860) "knew" about manifolds, algebraic varieties, and M_g (although these terms may not have been completely defined). Weyl (in 1913) defined manifolds through equivalence classes of atlases, Mumford (in 1965) proved that M_g exists, and Weil (in 1945) gave a definition of abstract algebraic varieties.
If $(X, \mathcal{O}_X)$ is a ringed space and $U \subset X$ is open, we have a ringed space $(U, \mathcal{O}_X|_U)$.

Example 2.2. Let $D \subset \mathbb{C}^n$ be open (in the classical topology) and $X \subset D$ be a set of zeros of a finite number of $\mathbb{C}$-analytic functions on D. Then for $U \subset X$ open:

$$\mathcal{O}_X(U) = \{f : U \to \mathbb{C} \mid \forall p \in U \exists p \in N \subseteq U \text{ and } g \text{ on } N \text{ such that } f|_N = g\}$$

Definition 2.3. A $\mathbb{C}$ **analytic space** is a ringed space $(X, \mathcal{O}_X)$ such that every point has a neighborhood U such that $(U, \mathcal{O}_X(U))$ is isomorphic as a ringed space to one of these.

This definition can be modified slightly for defining other spaces like C^∞-manifolds and $\mathbb{R}$-manifolds.
We have a sheaf $\mathcal{O}_X$ where for any open set $U \subset X$, $\mathcal{O}_X(U)$ is:

$$\{f : U \to k | \forall p \in U \exists N \ni p \text{ and } g, h \in k[x_1, \ldots, x_n], h(q) \neq 0 \text{ and } f(q) = \frac{g(q)}{h(q)} \forall q \in N\}$$

Definition 2.4. A **(Serre) algebraic variety**, not necessarily irreducible, is a ringed space $(X, \mathcal{O}_X)$ locally isomorphic to an affine algebraic set with the Zariski topology.

Example 2.5. Let A be a commutative ring with unit, $X = \text{Spec } A$ and $U \subset X$ open. Then $\mathcal{O}_X(U)$ is:

$$\{(f_p \in A_p)_{p \in U} \mid \forall p \in U \exists N \ni p \text{ and } g, h \in A \text{ such that } \forall q \in N, h \notin q, \frac{g}{h} \mapsto f_q \in A_q \forall q \in N\}$$

Exercise 2.6. $(X, \mathcal{O}_X)$ is a ringed space.

Definition 2.7. A **scheme** is a ringed space $(X, \mathcal{O}_X)$ that is locally isomorphic to one of these. An **affine scheme** is a scheme which is isomorphic to one of these.

More precisely,

Definition 2.8. An **affine scheme** is a locally ringed space $(X, \mathcal{O}_X)$ that is isomorphic as a locally ringed space to $(\text{Spec } A, \widetilde{A})$ where A is a commutative ring with unit and $\widetilde{A} = \mathcal{O}_{\text{Spec } A}$.

Remark 2.9. We need not include "locally" in the above definiition since $(\text{Spec } A, \widetilde{A})$ is a locally ringed space, however we include it in the definition so that we stay in the category of locally ringed spaces.

3. Gluing Construction

Let $\{(X_\alpha, \mathcal{O}_{X_\alpha})\}_{\alpha \in A}$ be a collection of ringed spaces, where A is an index set. We want to construct a topological space X from the X_α by gluing them together in some way. Suppose that for all pairs α, β we have open subsets $U_{\alpha\beta}$ of X_α (with induced ringed space structures $\mathcal{O}_{U_{\alpha\beta}} = \mathcal{O}_{X_\alpha}|_{U_{\alpha\beta}}$) with isomorphisms of ringed spaces:

$$(U_{\alpha\beta}, \mathcal{O}_{X_\alpha}|_{U_{\alpha\beta}}) \xrightarrow{\mathcal{O}_{\beta\alpha}} (U_{\beta\alpha}, \mathcal{O}_{X_\beta}|_{U_{\beta\alpha}})$$

Assume that $\forall \alpha, \beta, \Gamma \in A$:

- $U_{\alpha\alpha} = X_\alpha$
- $\mathcal{O}_{\alpha\alpha} =$Identity
- $\mathcal{O}_{\alpha\beta} = \mathcal{O}_{\beta\alpha}^{-1}$
- $U_{\alpha\beta} \cap U_{\alpha\Gamma} \underset{\cong}{\xrightarrow{\mathcal{O}_{\beta\alpha}}} U_{\beta\alpha} \cap U_{\beta\Gamma}$
- $\mathcal{O}_{\Gamma\alpha} = \mathcal{O}_{\Gamma\beta} \circ \mathcal{O}_{\beta\alpha}$ on $U_{\alpha\beta} \cap U_{\alpha\Gamma}$

Then one can construct a ringed space $(X, \mathcal{O}_X)$ with an open covering $\{U_\alpha\}$ of X with ringed space isomorphisms:

$$\varphi_\alpha : X_\alpha \xrightarrow{\sim} U_\alpha \text{ that satisfy:}$$

$$\varphi_\alpha(U_{\alpha\beta}) = U_\alpha \cap U_\beta \text{ and } \varphi_\beta \circ \mathcal{O}_{\beta\alpha} = \varphi_\alpha \text{ on } U_{\alpha\beta}$$

$$\begin{array}{ccc} U_{\alpha\beta} \subset X_\alpha & \xrightarrow{\varphi_\alpha} & X \\ \Big\downarrow \mathcal{O}_{\beta\alpha} & & \\ U_{\beta\alpha} \subset X_\beta & \xrightarrow{\varphi_\beta} & X \end{array}$$

This is unique up to canonical isomorphism. The topological space X is the quotient space of $\sqcup X_\alpha / \sim$ where the equivalence relation comes from the $\mathcal{O}_{\beta\alpha}$. U_α = image of X_α in X. Construct $\mathcal{O}_X$ so $\mathcal{O}_X|_{U_\alpha} \cong \mathcal{O}_{X_\alpha}$. If each X_α is a Serre varieties/$\mathbb{C}$-analytic space/scheme then X is as well.

Example 3.1. Let $X_1 = \mathbb{A}^1 = X_2, U_{12} = \mathbb{A}^1 - \{0\} = U_{21}$ and $U_{12} \xrightarrow{\mathcal{O}_{21}} U_{21}$:

- If $\mathcal{O}_{21}(x) = \frac{1}{x}$, then we construct X by identifying areas close to 0 in U_{12} with those close to ∞ in U_{21} and vice-versa so that $X \cong \mathbb{P}^1$.
- If $\mathcal{O}_{21}(x) = x$, then we get a "line with two origins" or the "bug-eyed-line."

Example 3.2. Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring so that $S_d \cdot S_e \subseteq S_{d+e}$. As a set, define:

$$\text{Proj }(S) = \{ \text{ homogeneous prime ideals } p \mid p \not\supseteq S_+ = \oplus_{d>0} S_d \}$$

Let $X = \text{Proj } S$. Let $\mathcal{A} \subseteq S$ be a homogenous ideal and define the closed sets in X to be of the form $V(\mathcal{A}) = \{q \in X \mid q \supseteq \mathcal{A}\}$. Given $p \in X$, we can localize S at the multiplicative set of homogeneous elements not in p, the ring is graded by $\mathbb{Z}$ through the degree of elements in S: $\deg(\frac{f}{g}) = \deg(f) - \deg(g)$. Let $S_{(p)}$ be set of elements in S_p of degree 0. For any open set $U \subseteq X$ we get a local ring where elements of the sheaf $\mathcal{O}_X(U)$ are:

$$\{(f_p \in S_{(p)})_{p \in U} | \forall p \in U \exists N \ni p \text{ and } d \in \mathbb{N}, g, h \in S_d, h \notin q \forall q \in N \text{ and } \frac{g}{h} \mapsto f_q \in S_{(q)} \forall q \in N\}$$

Lemma 3.3. *$(X, \mathcal{O}_X)$ is a scheme.*

Proof. For f in S_d, let $D(f) = \{p \in X \mid f \notin p\}$. Let $S_{(f)} \subseteq S_f$ be the subring of degree 0.
Claim: $D(f) \cong \text{Spec }(S_{(f)})$ as a ringed space. □

Lecture 6. January 27, 2009

Exercise 3.4. Hartshorne: §1:21 and §2: 3,5,7,8,9,10,12

Definition 3.5. Let $(R, m_R) \xrightarrow{\varphi} (S, m_S)$ be a ring homomorphism between the local rings R and S. Then φ is a **local ring homomorphism** if either of the following equivalent conditions hold:

(1) $\varphi^{-1}(m_S) = m_R$
(2) $\varphi(m_R) \subseteq m_S$

Proposition 3.6. *A map of rings $A \xrightarrow{\varphi} B$ induces a morphism of locally ringed spaces:*

$$(\text{Spec } B, \widetilde{B}) \xrightarrow{(f, f^\#)} (\text{Spec } A, \widetilde{A})$$

Proof. The homomorphism $A \xrightarrow{\varphi} B$ induces the continuous map $\text{Spec } B \xrightarrow{f} \text{Spec } A$ given by $Q \mapsto f(Q) = \varphi^{-1}(Q)$. We need to extend this to a local morphism of sheaves of rings $\widetilde{A} \longrightarrow f_*\widetilde{B}$. We want a map f_* that is compatible with restrictions and such that $\forall U \subseteq \text{Spec } A$, $\widetilde{A}(U) \longrightarrow f_*\widetilde{B}(U) = \widetilde{B}(f^{-1}(U))$. It is sufficient to check this for some open set in the basis $D(g) \subseteq \text{Spec } A$, where $g \in A$:

$$A[\frac{1}{g}] = \widetilde{A}(D(g)) \longrightarrow \widetilde{B}(f^{-1}(D(g))) = \widetilde{B}(D(\varphi(g)) = B[\frac{1}{\varphi(g)}]$$

$$\widetilde{A}(D(g)) \ni \frac{a}{g^t} \mapsto \frac{\varphi(a)}{\varphi(g^t)} = \frac{\varphi(a)}{(\varphi(g))^t} \in \widetilde{B}(f^{-1}(D(g))$$

Any principal open subset of $D(g)$ is of the form $D(gh)$ so this map is compatible with restriction. Now we need to check that this map is local:
Say $q \in \text{Spec } B, p = f(q)$. We need $(\widetilde{A})_p \longrightarrow (\widetilde{B})_q$ to be local. Since A, B can be identified with $\widetilde{A}, \widetilde{B}$, respectively, we can look at $A_p \longrightarrow B_p$ instead. Now (A_p, p) and (B_q, q) are local rings and $\varphi(p) = \varphi(f(q)) = \varphi(\varphi^{-1}(q)) \subseteq q$, therefore this is a local map. □

Theorem 3.7. *Every morphism of affine schemes, in the category of locally ringed spaces,*

$$(X, \mathcal{O}_X) \xrightarrow{(f,f^\#)} (Y, \mathcal{O}_Y),$$

where we identify X with Spec B and Y with Spec A, is induced by a unique map of rings $A \xrightarrow{\varphi} B$.

Caution: There are morphisms of ringed spaces $(\text{Spec } B, \widetilde{B}) \longrightarrow (\text{Spec } A, \widetilde{A})$ that are not induced by a map $A \xrightarrow{\varphi} B$ (e.g. p40, exercise 11 in Shaf. II).

Proof. Given $(\text{Spec } B, \widetilde{B}) \xrightarrow{(f,f^\#)} (\text{Spec } A, \widetilde{A})$, where $X = \text{Spec } B$ and $Y = \text{Spec } A$, so set $A = \mathcal{O}_Y(Y)$ and $B = \mathcal{O}_X(X)$. We need a map $A \longrightarrow B$. $\widetilde{A} \longrightarrow f_*\widetilde{B}$ by sending $\widetilde{A}(Y) \mapsto f_*\widetilde{B}(X) = \widetilde{B}(f^{-1}(Y)) = \widetilde{B}(X) = B$. Now check that the induced map Spec $B \longrightarrow$ Spec A is given by $Q \mapsto \varphi^{-1}(Q) = f(Q)$. In order to check this, we use the fact that this is a map of locally ringed spaces. Whenever $f(Q) = P$, $A_P = \mathcal{O}_{Y,P} \xrightarrow{f^\# = \varphi} \mathcal{O}_{X,Q} = B_Q$ is the local map: check that $\varphi^{-1}(QB_Q) = PA_P$ Therefore $\varphi^{-1}(Q) = P = f(Q)$. □

Example 3.8. $\mathbb{Z} \hookrightarrow \mathbb{Z}[x]$ gives us a map Spec $\mathbb{Z}[x] \xrightarrow{f}$ Spec $\mathbb{Z}$. Recall that the points in Spec $\mathbb{Z}$ are the prime ideals and Spec $\mathbb{Z}$ has one dimension. The points of Spec $\mathbb{Z}[x]$ are: $(0), (p), (g), (p, f)$ where p is prime, g is irreducible, and f is irreducible mod p. For instance over the point $(p) \in \mathbb{Z}$ we have closed points (p, f), where f is irreducible mod p, and non-closed points (p) and (f).
Let $Q = (2, x)$. Then $f(Q) = (2) = Q \cap \text{Spec } \mathbb{Z}$ and when we localize at these ideals we get a map $\mathbb{Z}_{(2)} \longrightarrow (\mathbb{Z}[x])_{(2,x)}$. However, in order to define this map we need to go through $f_*\mathbb{Z}[x]_{(2)}$. We get the commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}_{(2)} & \longrightarrow & (\mathbb{Z}[x])_{(2,x)} \\ \downarrow & \nearrow & \\ f_*\mathbb{Z}[x]_{(2)} & & \end{array}$$

$$\begin{aligned} f_*\mathbb{Z}[x]_{(2)} &= \varinjlim_{(2)\in U} \widetilde{\mathbb{Z}}[x](U) \\ &= \varinjlim_{(2)\in D(m)} \widetilde{\mathbb{Z}}[x](D(m)) \\ &= \varinjlim_{(2)\in D(m)} \mathbb{Z}[x][\frac{1}{m}] \\ &= \mathbb{Z}_{(2)}[x] \end{aligned}$$

Exercise 3.9. Explore $\mathbb{Z} \longrightarrow \mathbb{Z}[i]$ with Spec $\mathbb{Z}[i] \longrightarrow \mathbb{Z}$.

Example 3.10. Let X be a $\mathbb{C}$-manifold and $\mathcal{O}_X^{an}$ its sheaf of analytic functions. Then:

$$\mathcal{O}_X^{an} \longrightarrow (\mathcal{O}_X^{an})^* \text{ by sending } f \mapsto e^{2\pi i f}$$

is a sheaf of abelian groups where the group on the left is additive and the sheaf on the right is multiplicative. The kernel of this map is the locally constant sheaf $\underline{\mathbb{Z}}_X$ so that:

$$0 \longrightarrow \underline{\mathbb{Z}}_X \longrightarrow \mathcal{O}_X^{an} \longrightarrow (\mathcal{O}_X^{an})^* \text{ by sending } f \mapsto e^{2\pi i f} \text{ is exact}$$

Let $X = \mathbb{C}$ and $U = \mathbb{C} - \{0\}$. On $\mathbb{C}$, the logarithm can only be defined locally, but it cannot be defined on all of U. Therefore on U the map $\mathcal{O}_X^{an}(U) \longrightarrow (\mathcal{O}_X^{an})^*(U)$ is not surjective. Hence the cokernel presheaf is not zero. However, locally the exponential map has an inverse, and so if we restrict to a neighborhood $V \subset U$ on which the logarithm can be defined, then the map $\mathcal{O}_X^{an}(V) \longrightarrow (\mathcal{O}_X^{an})^*(V)$ is surjective. Hence the cokernel sheaf is zero.

Lecture 7. January 29, 2009

Definition 3.11. A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ which admits an open cover $\{(U_i, \mathcal{O}_X|_{U_i})\}$ by affine schemes.

Fix an $\mathbb{N}$-graded commutative ring with unit, $S = S_0 \oplus S_1 \oplus S_2 \oplus \dots$.
The **irrelevant ideal** of S is the ideal generated by positive degree, denote by S_+.
Recall: As a set, Proj $S = \{p \in \text{Spec } S \mid p \text{ is homogeneous } \} \setminus \mathbb{V}(S_+)$. Its topology is the Zariski topology induced by the subspace topology on Spec S. Closed sets all have the form: $\mathbb{V}(I)$ where I is homogenous (excluding the points in $\mathbb{V}(S_+)$).

Example 3.12. Let $S = k[x_0, \dots, x_n] = k \oplus [k[x_0, \dots, x_n]]_1 \oplus \dots = \bigoplus_{i \geq 0} [[k[x_0, \dots, x_n]]_i$.

Then: $S_+ = (x_0, \dots, x_n)$ and Proj $S = \{p \subset k[x_0, \dots, x_n] \text{ homogeneous} \mid (x_0, \dots x_n) \nsubseteq p\}$

Proj S is in 1-1 correspondence with the irreducible subvarieties of $\mathbb{P}^n$:
The point $[1 : 0 : \dots : 0] \in \mathbb{P}^n$ corresponds to the maximal ideal $(x_1, \dots, x_n) \in$ Proj S.
We get the following analogy, where $\mathbb{A}$ and $\mathbb{P}$ are viewed in classical algebraic geometry:
$\mathbb{A}_k^n$ corresponds to the closed points of Spec $k[x_1, \dots, x_n]$ with the subspace topology.
$\mathbb{P}_k^n$ corresponds to the closed points of Proj $k[x_0, \dots, x_n]$ with the subspace topology.

Let $S = \bigoplus_{i \geq 0} S_i$. A basis of open sets for Proj S is given by sets of the form

$$D_+(f) = \{p \in \text{Proj } S \mid f \notin p\} = D(f) \cap \text{Proj } S$$

, where f is a homogeneous polynomial of degree atleast 1 and $D(f) \subseteq$ Spec S.
$U = \text{Proj } S - \mathbb{V}(\{g_\lambda\}_{\lambda \in \Lambda}) = \cup_{\lambda \in \Lambda} D_+(g_\lambda)$, where the g_λ are homogeneous.

Example 3.13. $X = \text{Proj } k[x_0, \dots, k_n]$ is covered by $D_+(x_0) \cup \dots \cup D_+(x_n)$.
$\mathcal{O}_X(D_+(x_i)) = k[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] = k[x_0, \dots, x_n][\frac{1}{x_i}]_0$.

Definition 3.14. Proj S has a sheaf of rings $\widetilde{S}$ which has value

$$\widetilde{S}(D_+(f)) = \big[S[\frac{1}{f}]\big]_0 = \{\frac{S}{f^t} \mid \deg S = \deg f^t\}$$

Let $X = \text{Proj } S, \mathcal{O}_X = \widetilde{S}$. The definition is compatible with localization: $D_+(f) \supseteq D_+(gf)$.

$$\mathcal{O}_X(D_+(f)) \xrightarrow{\text{restriction}} \mathcal{O}_X(D_+(gf))$$

$$(S\Big[\frac{1}{f}\Big])_0 \xrightarrow{\text{localization}} (S\Big[\frac{1}{fg}\Big])_0 = (S\Big[\frac{1}{f}\Big]\Big[\frac{1}{g}\Big])_0$$

$\mathbb{P}_k^n = \text{Proj } k[x_0, \dots, x_n]$

$$\mathcal{O}_{\mathbb{P}_k^n}(D_+(x_i)) \longrightarrow \mathcal{O}_{\mathbb{P}_k^n}(D_+(x_i x_j))$$

Localization at $\frac{x_j}{x_i}$ induces the map:

$$\mathcal{O}_{\mathbb{P}^n_k}(D_+(x_i)) = k\big[\frac{x_0}{x_i},\dots,\frac{x_n}{x_i}\big] = k[x_0,\dots,x_n]\big[\frac{1}{x_i}\big]_0 \longrightarrow \big(k\big[\frac{x_0}{x_i},\dots,\frac{x_n}{x_i}\big]\big[\frac{1}{\frac{x_j}{x_i}}\big]\big)_0 = k[x_0,\dots,x_n][\frac{1}{x_i x_j}]_0$$

Recall that $D_+(f) \supseteq D_+(h) \Leftrightarrow h^n = gf$ for some n, g.

Stalks of $\mathcal{O}_{\text{Proj } S}$:
Let $(X, \mathcal{O}_X) = (\text{Proj } S, \widetilde{S})$. Take any $p \in X$.

$$\mathcal{O}_{X,p} = \varinjlim_{U \ni p} \mathcal{O}_X(U) = \varinjlim_{D_+(f)\ni p} \mathcal{O}_X(D_+(f)) = \varinjlim_{f \notin p} \big[S\big[\frac{1}{f}\big]\big]_0 = [S[T^{-1}]]_0 \equiv S_{<p>}$$

Where all of the f's above are homogeneous and $T \subseteq S$ is the multiplicatve system in S of homogeneous elements not in p. Every element has the form $\frac{a}{f^t}$ where $a \in S, f \notin p$ and the $\deg a = \deg f^t$. Then $\frac{a}{f^t} \in$ unique maximal ideal $\Leftrightarrow a \in p$.

Let $X = \text{Proj } k[x_0,\dots,x_n] \ni p = (x_1,\dots,x_n)$. All of the following f are homogeneous.

$$\begin{aligned}
\mathcal{O}_{X,p} &= \varinjlim_{p\in D_+(f)} \mathcal{O}_X(D_+(f)) \\
&= \varinjlim_{p\in D_+(fx_0)} \mathcal{O}_X(D_+(fx_0)) \\
&= \varinjlim_{fx_0 \notin p} k\left[\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right]\left[\frac{1}{\left(\frac{f}{x_0^{\deg f}}\right)}\right] \\
&= k\left[\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right]_{(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0})} \\
&= \mathcal{O}_{\mathbb{P}^n_k,p}
\end{aligned}$$

Theorem 3.15. *$(\text{Proj } S, \mathcal{O}_{\text{Proj } S}) = (\text{Proj } S, \widetilde{S})$ is a scheme.*

Proof. Recall that Proj S is covered by open sets $D_+(f)$ where $f \in S_+$.

$$(D_+(f), \widetilde{S}|_{D_+(f)}) \xrightarrow{\varphi,\varphi^{\#}} (\text{Spec } (S\big[\frac{1}{f}\big]_0) = A, \widetilde{A})$$

Check the details of this proof. □

Remark 3.16. Proj is not a functor from graded rings to schemes. For example consider $k[x,y] \hookrightarrow k[x,y,z]$. Then the map on Proj would be: Proj $k[x,y] \longleftarrow$ Proj $k[x,y,z]$ when $(x,y) \longleftarrow (x,y)$, but (x,y) is not a point in Proj $k[x,y]$ since it is contained in the irrelevant ideal of $k[x,y]$.

Properties of Schemes:
Topological: connected, irreducible
Ring Theoretic: reduced, integral, (locally) Noetherian, (locally) finite type, finite

Definition 3.17. A scheme is **connected** if its corresponding topological space is connected.

Definition 3.18. A scheme $(X, \mathcal{O}_X)$ is **reduced (locally integral, locally Noetherian)** if X has a cover U_i by open affine sets such that each $\mathcal{O}_X(U_i)$ is reduced (domain, Noetherian).

Proposition 3.19. *Equivalently, every open affine set will have the property given by the above definitions.*

Proposition 3.20. *A scheme is integral $\Leftrightarrow$ it is reduced and irreducible.*

Remark 3.21. If a ring R is Noetherian, then Spec R is Noetherian. However if Spec R is Noetherian, it need not be true that R is Noetherian. For instance, consider $\mathbb{Z}^+$ =integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$. $\mathbb{Z}^+$ is definitely not Noetherian since the chain $(2) \subset (2^{\frac{1}{2}}) \subset (2^{\frac{1}{2^2}}) \subset \ldots$ does not terminate. Spec $\mathbb{Z}^+$ is not a Noetherian scheme, but as a topological space it is Noetherian since every descending chain of closed sets stabilizes.

Lecture 8. February 3, 2009

Exercise 3.22. 2.4, 2.15, 2.18, 3.3, 3.6, 3.8, 3.10, 3.12. Read Hartshorne II §1,2,3 and Shaf. V through 4.1.

Properties of Schemes:

Definition 3.23. An A**-algebra** is a a ring map $A \longrightarrow B$.

Definition 3.24. A scheme X is an S**-scheme** if there is a scheme map $X \longrightarrow S$. In this case we say that "X is a scheme over S."

Definition 3.25. An A-algebra B is of **finite-type** if it is finitely generated as an A-algebra. Equivalently, B has a presentation $B \cong \frac{A[x_1,\ldots,x_n]}{\sim}$.

Definition 3.26. An A-algebra is **finite** (or module finite) if B is finitely generated as an A-module.

Remark 3.27. Finite type and integral imply module finite.

Definition 3.28. An S-scheme $X \xrightarrow{f} S$ is of **locally finite type** if S has an open affine cover $S = \cup_i V_i$ such that each $f^{-1}(V_i)$ has an open affine cover $\cup_j U_{ij}$ such that the induced map of affine schemes

$$U_{ij} = \text{Spec } (B_{ij}) \xrightarrow{f|_{U_{ij}}} V_i = \text{Spec } A_i$$

comes from a finite type algebra $A_i \longrightarrow B_{ij} \forall i, j$. Furthermore, this S-scheme is of **finite type** if the cover $\cup_j U_{ij}$ of each $f^{-1}(V_i)$ can be taken to be a finite cover.

Example 3.29. Spec $k[x_1, \ldots, x_n] \longrightarrow$ Spec k is given by the finite type map $k \hookrightarrow k[x_1, \ldots, x_n]$.

Example 3.30. Proj $k[x_0, \ldots, x_n] \xrightarrow{f}$ Spec k has finite type:
$D_+(x_i) =$ Spec $k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}] \longrightarrow$ Spec k is given by the map $k \hookrightarrow k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$. Spec $k = \{(0)\}$ so (0) is an open cover and $f^{-1}((0)) =$ Proj $k[x_0, \ldots, x_]$, which has a finite cover given by $\{D_+(x_i)\}i = 1^n$. Therefore f is of finite type.

Definition 3.31. An S-scheme is **finite** if S has an affine cover $S = \cup_i V_i$ such that EACH $f^{-1}(V_i) = U_i$ is affine, and the induced map $U_i =$ Spec $B_i \longrightarrow V_i =$ Spec A_i of affine schemes corresponds to a finite map $A_i \longrightarrow B_i$.

Example 3.32. $k[t] \hookrightarrow \frac{k[t,x]}{(x^2-t)} \cong k[t] \oplus xk[t]$ is a module finite map. This induces the map Spec $\frac{k[t,x]}{(x^2-t)} \longrightarrow$ Spec $k[t]$ so that the point $(x - \lambda, t - \lambda^2) \mapsto (t - \lambda^2)$. This is a finite map of schemes.

Example 3.33. Given $k[t, s] \hookrightarrow \frac{k[t,s,x]}{x^2-ts}$, where $s \mapsto s$ and $t \mapsto t$, is there an induced map:

$$\text{Proj } \frac{k[t, s, x]}{x^2 - ts} \xrightarrow{f} \text{Proj } k[t, s] \text{ ?}$$

Spec $k[\frac{t}{s}] = D_+(s)$ and Spec $k[\frac{s}{t}] = D_+(t)$ in Proj $k[t, s]$ so that

$$f^{-1}(D_+(s)) = D_+(s) = \text{Spec } \frac{k[\frac{t}{s}, \frac{x}{s}]}{(\frac{x}{s})^2 - (\frac{t}{s})}$$

This is induced by the module finite map: $k[\frac{t}{s}] \hookrightarrow \frac{k[\frac{t}{s}, \frac{x}{s}]}{(\frac{x}{s})^2-(\frac{t}{s})}$.

Definition 3.34. An **open subscheme** of a scheme X (i.e. $(X, \mathcal{O}_X)$) is an open set $U \subset X$ with the sheaf given by $\mathcal{O}_X|_U$.

Definition 3.35. A **closed subscheme** of a scheme X is scheme $(Y, \mathcal{O}_Y)$, where $Y \subseteq^i X$ as a closed set, together with a surjective map of sheaves of rings $\mathcal{O}_X \longrightarrow i_* \mathcal{O}_Y$, i.e. $\forall U \subseteq X$, $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_Y(U \cap Y) = \mathcal{O}_Y(i^{-1}(U))$ is surjective.

Example 3.36. Let $X = \text{Spec } k[x,y] = \mathbb{A}^2_k$

$$k[x,y] \twoheadrightarrow \frac{k[x,y]}{x^3} \twoheadrightarrow \frac{k[x,y]}{x^2} \twoheadrightarrow \frac{k[x,y]}{x} \twoheadrightarrow \frac{k[x,y]}{(x,y)}$$

Induces maps on schemes:

$$\mathbb{A}^2_k = \text{Spec } k[x,y] \longleftarrow \text{Spec } \frac{k[x,y]}{x^3} \longleftarrow \text{Spec } \frac{k[x,y]}{x^2} \longleftarrow \text{Spec } \frac{k[x,y]}{x} \longleftarrow \text{Spec } \frac{k[x,y]}{(x,y)}$$

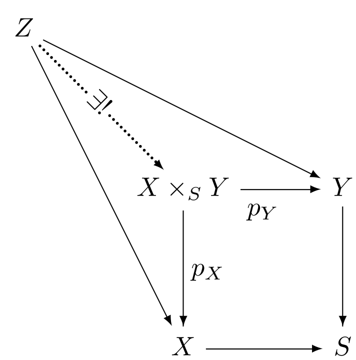
where each is a map of closed subschemes.
Moral: There are usually many closed subschemes of a given scheme, and they can have the same underlying topological space, but be different schemes. This is not true for open subschemes.

Remark 3.37. There is a unique smallest closed subscheme of a given reduced scheme X with support a given closed set. This is called the reduced subscheme of X supported at Y. Subvarieties correspond to reduced subschemes of varieties. In the previous example, the smallest closed subscheme is Spec $\frac{k[x,y]}{(x,y)}$.

4. Products

We want to generalize products from classical algebraic geometry: $\mathbb{A}^m_k \times_k \mathbb{A}^n_k = \mathbb{A}^{n+m}_k$.

Definition 4.1. Let $X \longrightarrow S$ and $Y \longrightarrow S$ be S-schemes. The **product** is an S-scheme denoted $X \times_S Y$ together with morphisms $p_X : X \times_S Y \longrightarrow X$ and $p_Y : X \times_S Y \longrightarrow Y$ that satisfy the following universal property (in the category of S-schemes): Given Z such that $Z \longrightarrow X$ and $Z \longrightarrow Y$, there exists a unique map from Z to the product which makes the diagram commute.



In the category of sets: Given $X \xrightarrow{f} S$ and $Y \xrightarrow{g} S$, then the equivalent product in sets (fibered product) would be:

$$X \times_S Y = \{(x,y) \in X \times Y \mid f(x) = g(y)\}$$

Remark 4.2. If you see $X \times Y$ without reference to a base scheme S, then we must rely on the context to determine S. If there is no such context, then it could mean that every scheme is a Spec $\mathbb{Z}$-scheme in a unique way so $X \times Y$ could mean $X \times_{\text{Spec } \mathbb{Z}} Y$. More often, X and Y are schemes over some unnamed base scheme. Also, sometimes people write $X \times_k Y$ instead of $X \times_{\text{Spec } k} Y$ for short-hand.

Theorem 4.3. *Products exist for S-schemes.*

Proof. **Affine Case:** $X = \text{Spec } B \longrightarrow S = \text{Spec } T$ and $Y = \text{Spec } A \longrightarrow S = \text{Spec } T$, then $X \times_S Y = \text{Spec } (A \otimes_T B)$.
Suppose that $Z = \text{Spec } R$ is a scheme and $A \longrightarrow Z$, $B \longrightarrow Z$. Then there exists a unique map $A \otimes_T B \longrightarrow Z$ such that the diagram commutes.

$$\begin{array}{ccc} A \otimes_T B & \longleftarrow & B \\ \uparrow & & \uparrow \\ A & \longleftarrow & T \end{array}$$

The property for schemes follows for affine Z immediately. By exercise 2.4 in Hartshorne, to give a scheme map $Z \longrightarrow \text{Spec } (A \otimes_T B)$ is equivalent to giving a ring map $A \otimes_T B \longrightarrow \mathcal{O}_Z(Z)$.
General Case: In general, $X = \cup \text{Spec } B_{ik}, Y = \cup \text{Spec } A_{ij}$ and $S = \text{Spec } T_i$. Then define $W_{ikj} = \text{Spec } (A_{ik} \otimes_{T_i} B_{ij})$. We can glue these together to get $X \times_S Y$.
Read Hartshorne for more details. □

Recall: For a point $p \in Y$, $k(p)$ is the residue field of p and $\text{Spec } k(p) \longrightarrow Y$.

Definition 4.4. Given a morphism $X \xrightarrow{f} Y$ of schemes, and a point $p \in Y$, the **(scheme-theoretic) fiber** of f over p is the scheme: $\text{Spec } k(p) \times_Y X$.

This agrees with the definition of fiber for topological spaces, but it also has a scheme structure

Example 4.5. $k[t] \hookrightarrow \frac{k[t,x]}{x^2-t}$, where k is algebraically closed.

$$\text{Spec } \frac{k[t,x]}{x^2-t} \longrightarrow \text{Spec } (k[t])$$

Take the point $p = (t - \lambda)$

$$k[t] \longrightarrow k(p) = \frac{k[t]}{t-\lambda} \text{ so that Spec } k(p) \longrightarrow \text{Spec } (k[t]).$$

The fiber over p is:

$$\begin{aligned} \text{Spec } \left(\frac{k[t]}{t-\lambda} \otimes_{k[t]} \frac{k[t,x]}{x^2-t}\right) &= \text{Spec } \frac{\frac{k[t]}{t-\lambda}[x]}{x^2-t} \\ &= \text{Spec } \frac{k[x]}{x^2-\lambda} \\ &= \text{Spec } \frac{k[x]}{(x-\sqrt{\lambda})(x+\sqrt{\lambda})} \\ &= \text{Spec } \frac{k[x]}{x-\sqrt{\lambda}} \oplus \frac{k[x]}{x+\sqrt{\lambda}} \end{aligned}$$

The fiber over (t) is: $\text{Spec } \frac{k[x]}{(x^2)}$.

Lecture 9. February 5, 2009

New policy: Choose one problem of the assigned to do with one other person and turn in one carefully written paper which is proofread by both. (In addition to other probems written up individually).

Example 4.6. Let $X = \text{Spec } \mathbb{C}[x]$ and $Y = \text{Spec } \mathbb{C}[y]$. Then

$$X \times_{\text{Spec } \mathbb{C}} Y = \text{Spec } (\mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[y]) = \text{Spec } \mathbb{C}[x, y]$$

Uses of the Product:

(1) Defining fibers of $X \xrightarrow{f} Y$
(2) Base change
(3) Defining families of schemes or varieties
(4) Deformations of varieties or schemes
(5) Defining separatedness, a property of schemes analogous to the Hausdorff property for topological spaces.

Definition 4.7. The (scheme-theoretic) **fiber** of a map $X \xrightarrow{f} Y$ over a point $p \in Y$ is $\text{Spec } k(p) \times_Y X$.

In the category of sets, this is literally the fiber.

Example 4.8. Given a map of affines schemes $\text{Spec } B \xrightarrow{f} \text{Spec } A$, the corresponding map on rings $A \xrightarrow{\varphi} B$, and a point $p \in \text{Spec } A$, then:

$$f^{-1}(p) = \text{Spec } \big(\frac{A_p}{pA_p} \otimes_A B\big) \subseteq \text{Spec } B$$

Let's see why this is a subset of Spec B.

$$\begin{aligned} \frac{A_p}{pA_p} \otimes_A B &= A\big([A-p]^{-1} \otimes_A \frac{A}{p}\big) \otimes_A B \\ &= A[(A-p)^{-1}] \otimes_A \frac{B}{pB} \\ &= \frac{B}{pB}[[\varphi(A-p)]^{-1}] \end{aligned}$$

The prime ideals in $\frac{B}{pB}[[\varphi(A-p)]^{-1}]$ are in one-to-one correspondence with $q \in \text{Spec } B$ which satisfy $q \supseteq pB$ and are disjoint from U. This is in one-to-one correspondence with $\varphi^{-1}(q) \subseteq p$ and $\varphi^{-1}(q) = p$ which is in one to one correspondence with $q \in \text{Spec } B$ satisfying $\varphi^{-1}(q) = p$.

Base Change:
Given a scheme X over S an a morphism $S' \longrightarrow S$, there is a scheme X' over S' which people called the "S'-scheme obtained by base change," $S' \times_S X \longrightarrow S'$.

Example 4.9. $X_{\mathbb{R}} = \text{Spec } \frac{\mathbb{R}[x,y]}{x^2+y^2}$ is an affine, integral, finite type (but not finite) scheme of dimension one (this is the Krull dimension). Base change from $\mathbb{R}$ to $\mathbb{C}$:

$$X_{\mathbb{C}} = \text{Spec } \mathbb{C} \times_{\text{Spec } \mathbb{C}} X_{\mathbb{R}} = \text{Spec } \big(\mathbb{C} \otimes_{\mathbb{R}} \frac{\mathbb{R}[x,y]}{x^2+y^2}\big) = \text{Spec } \frac{\mathbb{C}[x,y]}{x^2+y^2} = \text{Spec } \frac{\mathbb{C}[x,y]}{(x+iy)(x-iy)}$$

This scheme still is of finite type and has dimension one, but it is not integral since $x^2 + y^2$ factors.

Finite type is preserved by base change,
Families:

Definition 4.10. A **family of schemes** parametrized by a scheme S is a morphism $X \longrightarrow S$. **Members** of the family are the fibers: $\{X_p\}_{p \in S}$, where $X_p = \text{Spec } k(p) \times_S X$.

Example 4.11. Let $k=\overline{k}$ and $k[t] \hookrightarrow \frac{k[t,x,y]}{xy-t}$ and so $\mathbb{A}^1_k = \text{Spec } k[t] \xleftarrow{f} \text{Spec } \frac{k[t,x,y]}{xy-t}$. This is a family of hyperbolas parametrized by $\mathbb{A}^1_k$. The members of the family are: Take any $\lambda \in \mathbb{A}^1_k$. Then

$$f^{-1}((t-\lambda)) = \text{Spec } (\frac{k[t]}{(t-\lambda)} \otimes_{k[t]} \frac{k[t,x,y]}{xy-t}) = \text{Spec } (\frac{k[x,y]}{xy-\lambda}$$

The generic fiber (over the generic point (0)) is:

$$f^{-1}((0)) = \text{Spec } \big(k(t) \otimes_{k[t]} \frac{k[t,x,y]}{xy-t}\big)$$

Let $L=k(t)$, then $f^{-1}((0)) = \text{Spec } \big(\frac{L[x,y]}{xy-t}\big)$.

Example 4.12. Consider Spec $\big(\frac{\mathbb{Q}[x,y,z]}{x^n+y^n-z^n}\big)$. We want to study the family that comes out of $\mathbb{Z} \hookrightarrow \frac{\mathbb{Z}[x,y,z]}{x^n+y^n-z^n}$. The family is given by:

$$\text{Spec } (\frac{\mathbb{Z}[x,y,z]}{x^n+y^n-z^n}) \xrightarrow{f} \text{Spec } \mathbb{Z}$$

Members of the family:
Closed fibers: $f^{-1}((p)) = \text{Spec } (\frac{\mathbb{Z}}{p\mathbb{Z}} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}[x,y,z]}{x^n+y^n-z^n}) = \text{Spec } \big(\frac{\mathbb{Z}_p[x,y,z]}{x^n+y^n-z^n}\big)$.
The fiber over the generic point (0) is:

$$f^{-1}((0)) = \text{Spec } \big(\mathbb{Q} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}[x,y,z]}{x^n+y^n-z^n}\big) = \text{Spec } \big(\frac{\mathbb{Q}[x,y,z]}{x^n+y^n-z^n}\big)$$

Part 2. Expanding our knowledge of Schemes

5. Properties of Schemes not derived from Rings

These properties come from the way in which the affine patches are glued together.
There are two main properties to consider:
separatedness: which corresponds to "Hausdorff" properties
properness: which corresponds to "compactness"

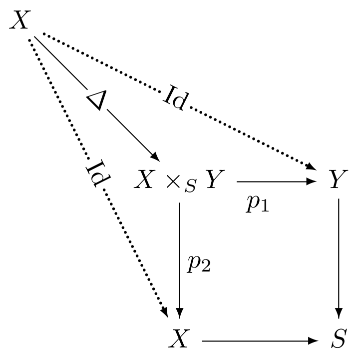
Examples of Separated morphisms of schemes $X \longrightarrow S$:

- Spec R is separated over Spec $\mathbb{Z}$ - so the morphism Spec $R \longrightarrow$ Spec $\mathbb{Z}$ is separated
- Any map of affine schemes, Spec $B \longrightarrow$ Spec A, is separated
- $S = S_0 \oplus S_1 \oplus \dots$, S_0 is a subring and Proj $S \longrightarrow$ Spec S_0 is separated
- Quasi-projective varieties are separated over k

Non-separated scheme:
The "bug-eyed line" over k is not separated over k. Spec $k[y,\frac{1}{y}] \longrightarrow$ Spec $k[x,\frac{1}{x}]$ given by: $k[y,\frac{1}{y}] \longleftarrow k[x,\frac{1}{x}]$ where $y \longleftarrow x$.

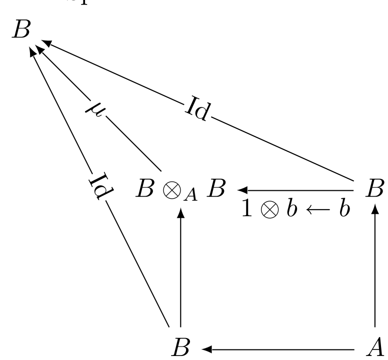
Proposition 5.1. *A topological space X is Hausdorff $\Leftrightarrow$ the diagonal $X \longrightarrow X \times X$ is a closed embedding of topological space.*

Definition 5.2. Given a scheme X over S, $X \longrightarrow S$, the **diagonal morphism** is the unique map $X \xrightarrow{\Delta} X \times_S X$ such that composition with projection onto X is the identity map.



Definition 5.3. $X \longrightarrow S$ is **separated** if the diagonal morphism is a closed embedding of schemes.

Example 5.4. Given Spec $B \longrightarrow$ Spec A



Let Spec $B \xrightarrow{\Delta = \text{Spec } \mu}$ Spec $(B \otimes_A B) =$ Spec $B \times_{\text{Spec } A}$ Spec B. This is obviously a closed embedding since $B \otimes_A B \xrightarrow{\mu} B$ is surjective.

Lecture 10. February 10, 2009

Exercise 5.5. Hartshorne II: §1 8,10,12,13,15 and §4 1,3,5ab.

Remark 5.6. When Hartshorne uses the term variety, he is referring to an irreducible, quasi-projective variety.

Definition 5.7. A morphism of schemes $X \xrightarrow{f} Y$ of finite type over $\mathbb{C}$ is **proper** if, in the Euclidean topology induced by $\mathbb{C}$, this map is proper (i.e. the preimage of a compact set is compact).

Definition 5.8. A morphism of schemes $X \xrightarrow{f} Y$ is **proper** if it is finite type, separated, and universally closed.

Example 5.9. Identity morphism, finite maps, closed embeddings, $\mathbb{P}^1_k \longrightarrow$ Spec k.

Definition 5.10. A map $X \longrightarrow Y$ is universally closed if it is closed, and remains closed under base change (i.e. given $Y' \longrightarrow Y$, the induced map $X \times_Y Y' \longrightarrow Y'$ is closed).

Theorem 5.11 (Theorem from 631)**.** *If X is a projective variety, then given any variety Y', the map $X \times Y' \longrightarrow Y'$ is closed.*

In scheme language this corresponds to: Let $X \longrightarrow$ Spec k projective and $Y' \longrightarrow$ Spec k, then $X \times_{\text{Spec } k} Y' \longrightarrow Y'$ is closed.

Example 5.12. Is $\mathbb{A}^1_k = \text{Spec } k[y] \longrightarrow \text{Spec } k$ is proper? We would expect not, since $\mathbb{A}^1_k$ is not compact. This morphism is of finite type and separated, so if we are right then it cannot be universally closed. Consider:

$$\mathbb{A}^2 \xrightarrow{\pi} \mathbb{A}^1 \quad (\lambda, \mu) \mapsto \lambda \qquad \text{Spec } k[x,y] \longrightarrow \text{Spec } k[x] \text{ induced by } x \longleftarrow x$$

The closed set $\mathbb{V}(xy-1) \subseteq \mathbb{A}^2 \xrightarrow{\pi} \mathbb{A}^1 - \{0\}$ and $\text{Spec } k[x,y] \supseteq \mathbb{V}(xy-1) \longrightarrow (\text{Spec } k[x] - (x))$.

$$\text{Spec } k[y] \times_{\text{Spec } k} \text{Spec } k[x] \longrightarrow \text{Spec } k[x]$$

So this is not universally closed $\Rightarrow$ not proper.

Valuation Rings (Review)

Definition 5.13. Fix an ordered abelian group G (usually $G = \mathbb{Z}$). A **G-valued valuation** on a field K (where $K^* = K \setminus \{0\}$) is a group homomorphism $\nu : K^* \longrightarrow G$ satisfying:

- $\nu(a) = \infty \Leftrightarrow a = 0$
- $\nu(ab) = \nu(a) + \nu(b), \forall a, b \in K^*$
- $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}, \forall a, b \in K^*$

The valuation ring of this valuation is $R_\nu = \{f \in K \mid \nu(f) \geq 0\}$. This is local with maximal ideal $m_\nu = \{f \in K \mid \nu(f) > 0\}$.
Equivalently, R is a **valuation ring** inside its fraction field $K \Leftrightarrow \forall x \in K$, either x or $x^{-1} \in R$.

Example 5.14. p-adic evaluations, the order of vanishing of a divisor

Definition 5.15. A valuation ring is a domain V which is the valuation ring for some valuation ν on its fraction field.

Definition 5.16. The valuation is discrete if $G = \mathbb{Z}$ and in this case the valuation ring is a **discrete valuation ring** (DVR).

CAUTION: Valuation rings are rarely Noetherian

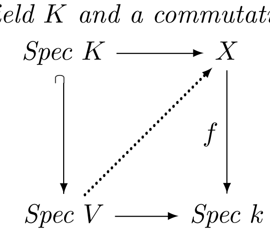
Theorem 5.17. *A Noetherian local domain is a DVR $\Leftrightarrow$ one of the following equivalent conditions hold:*

- *PID*
- *Normal of dimension 1*
- *The maximal ideal is principal*

Fix a DVR V with fraction field k, what does Spec V look like? V has dimension one and one maximal ideal m, so Spec V is a topological space with only two points: a closed point (m) and a generic point $\{\eta\} = \text{Spec } k = \text{Spec } V - \{m\}$, which is an open set of Spec V.

5.0.1. *Valuative Criterion for Separatedness/Properness:*

Theorem 5.18. *Let $X \xrightarrow{f} \text{Spec } k$ be a finite type scheme over k. Then f is separated if and only if the following criterion holds:*
Given any DVR V with fraction field K and a commutative diagram:



there is at most one way to map Spec $V \longrightarrow X$ and make the diagram commute.

Intuitively: a scheme over k is proper if and only if "it contains no bug-eyed lines." If a scheme over k contained a "bug-eyed line," then there would be more than one way to give a map Spec $V \longrightarrow X$.

Theorem 5.19. *Let $X \xrightarrow{f}$ Spec k be a finite type scheme over k. Then f is proper if and only if the following criterion holds:*
Given any DVR V with fraction field K and a commutative diagram:

$$\begin{array}{ccc} \text{Spec } K & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \text{Spec } V & \longrightarrow & \text{Spec } k \end{array}$$

there is exactly one way to map Spec $V \longrightarrow X$ and make the diagram commute.

Intuitively, a scheme over k is proper if it "has no holes."

Theorem 5.20. *A morphism $X \xrightarrow{f} Y$ of finite type, with X and Y Noetherian, is separated (respectively proper) if and only if given any discrete valuation ring V with fraction field K and a diagram:*

$$\begin{array}{ccc} \text{Spec } K & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \text{Spec } V & \longrightarrow & Y \end{array}$$

there is at most (respectively, exactly one) map h filling in the diagram.

Proof. This is very difficult so we are not going to cover it, but it can be found in Hartshorne. □

Intuitively, a scheme over k is separated if and only if it contain no "bug-eyed" lines.

Corollary 5.21. *Assume that all schemes below are Noetherian:*

(1) *open and closed immersions are separated, closed immersions are proper*
(2) *compositions of separated (respectively proper) morphisms are separated (respectively proper)*
(3) *separated (respectively proper) morphisms are stable under base change (i.e. $X \xrightarrow{f} Y$ is separated and $Y' \longrightarrow Y$, then $X \times_Y Y' \longrightarrow Y'$ is separated)*
(4) *products of separated morphisms (respectively proper) are separated (respectively proper)*
(5) *$X \xrightarrow{f} Y$ is separated (respectively proper) if Y has an open cover $\{U_i\}$ and each $f^{-1}(U_i) \longrightarrow U_i$ is separated (respectively proper)*

Of (5).

$$\begin{array}{ccccc} \text{Spec } k & \longrightarrow & f^{-1}(U_i) & \overset{\subseteq}{\hookrightarrow} & X \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec } V & \longrightarrow & U_i & \overset{\subseteq}{\hookrightarrow} & Y \end{array}$$

□

Lecture 11. February 12, 2009

Definition 5.22. Projective space over a scheme Y, denoted $\mathbb{P}^n_Y$, is $Y \times_{\text{Spec } \mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$, where $\mathbb{P}^n_{\mathbb{Z}} = \text{Proj } \mathbb{Z}[x_0, \ldots, x_n]$.

Example 5.23. If $Y = \text{Spec } A$, then

$$\begin{aligned}\mathbb{P}^n_Y &= \text{Spec } A \times_{\text{Spec } \mathbb{Z}} \text{Proj } \mathbb{Z}[x_0, \ldots, x_n] \\ &= \cup_{i=0}^n \text{Spec } A \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{Z}[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}] \\ &= \cup_{i=0}^n \text{Spec } (A \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}] \\ &= \cup_{i=0}^n D_+(x_i) \subseteq \text{Proj } A[x_0, \ldots, x_n] \\ &= \mathbb{P}^n_A\end{aligned}$$

Definition 5.24. A morphism of sheaves $X \xrightarrow{f} Y$ is **projective** if it factors as:

$$X \xhookrightarrow{\text{closed immersion}} Y \times_{\mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}} \xrightarrow{\pi_Y} Y$$

Example 5.25. Let $X = \mathbb{V}(F_0, \ldots, F_c) \subseteq \text{Proj } A[x_0, \ldots, x_n] \longrightarrow \text{Spec } A$, where the F_i are homogeneous polynomials in $x_0, \ldots, x_n$ with coefficients in A. Then the morphism $X \longrightarrow \text{Spec } A$ is projective.

Example 5.26. Let $A = \mathbb{C}[t_0, t_1]$. $X = \mathbb{V}(t_0 x_1 - t_1 x_0) \subseteq \text{Proj } A[x_0, x_1] \longrightarrow \text{Spec } \mathbb{C}[t_0, t_1] = \mathbb{A}^2_{\mathbb{C}}$

$$\text{Proj } A[x_0, x_1] = \text{Spec } A \times_{\text{Spec } \mathbb{C}} \text{Proj } \mathbb{C}[x_0, x_1] = \text{Spec } A \times_{\text{Spec } \mathbb{C}} \text{Proj } \mathbb{C}[x_0, x_1]$$

So that $X \subseteq \mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, where $\mathbb{A}^2_{\mathbb{C}}$ has coordinates t_0, t_1 and $\mathbb{P}^1_{\mathbb{C}}$ has coordinates $x_0 : x_1$. Look in the affine patch $\mathbb{A}^2_{\mathbb{C}} \times D_+(x_0)$.

$$\mathbb{V}(t_0 \frac{x_1}{x_0} - t_1) \subseteq \text{Spec } \mathbb{C}[t_0, t_1] \otimes \text{Spec } \mathbb{C}[\frac{x_1}{x_0}] = \text{Spec } k[t_0, t_1, \frac{x_1}{x_0}] \longrightarrow \text{Spec } \mathbb{C}[t_0, t_1]$$

where the map is given by inclusion of rings. Note that $\mathbb{V}(t_0 \frac{x_1}{x_0} - t_1) = \text{Spec } \frac{\mathbb{C}[t_0, t_1, \frac{x_1}{x_0}]}{(t_0 \frac{x_1}{x_0} - t_1)}$ and this has closed points:

- $(t_0 - \lambda_0, t_1 - \lambda_1, \frac{x_1}{x_0} - \mu)$, where $\lambda_0 \mu = \lambda_1$
- $(\lambda_0, \lambda_1, \frac{\lambda_1}{\lambda_0})$ if $\lambda_0 \neq 0$
- $(0, 0, \mu)$

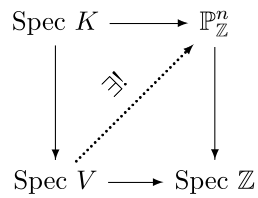
The map given by inclusion sends:
$(t_0 - \lambda_0, t_1 - \lambda_1, \frac{x_1}{x_0} - \mu) \mapsto (t_0 - \lambda_0, t_1 - \lambda_1)$
$(\lambda_0, \lambda_1, \frac{\lambda_1}{\lambda_0}) \mapsto (\lambda_0, \lambda_1)$
$(0, 0, \mu) \mapsto (0, 0)$
This is the schemified version of the classic blow-up at the origin.

Remark 5.27. Note that $\mathbb{C}[t] \otimes_{\mathbb{Z}} \mathbb{Z}[x] = \mathbb{C}[t] \otimes_{\mathbb{C}} \mathbb{C}[x]$

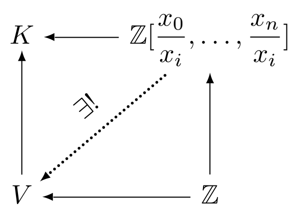
Theorem 5.28. *Projective morphisms are proper.*

Proof. Let $X \xhookrightarrow{\text{closed embedding}} Y \times \mathbb{P}^n_{\mathbb{Z}} \xrightarrow{\pi_Y} Y$ be a projective morphism, f.
Since closed embeddings are proper and compositions of proper morphisms are proper, it suffices to show that π_Y is proper. The map $Y \times \mathbb{P}^n_{\mathbb{Z}} \xrightarrow{\pi_Y} Y$ is a base change from $\mathbb{P}^n_{\mathbb{Z}} \longrightarrow \text{Spec } \mathbb{Z}$. Since properness is preserved by base change, it suffices to show that $\mathbb{P}^n_{\mathbb{Z}} \longrightarrow \text{Spec } \mathbb{Z}$ is proper. Instead of doing a base change we could do this by covering Y by affine sets $\{U_i\}$ and showing that the map $U_i \times \mathbb{P}^n_{\mathbb{Z}} \longrightarrow U_i$ is proper - this allows us to

assume that Y is affine.
Use the valuative criterion for properness:
$\mathbb{P}^n_{\mathbb{Z}} \longrightarrow \text{Spec } \mathbb{Z}$ is of finite type since $\mathbb{P}^n_{\mathbb{Z}}$ has affine open covers given by $D_+(x_i) = \text{Spec } \mathbb{Z}[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$ and the map from any open set in the cover to Spec $\mathbb{Z}$ has finite type.
Take any valuative ring V and fraction field K with valuation ν.



Let η be the image of the dense point $(0) \in \text{Spec } K$ in $\mathbb{P}^n_{\mathbb{Z}}$. Without loss of generality, we can assume that $\eta \in \cap_{i=0}^n D_+(x_i)$ since otherwise we would have that $\eta \in \mathbb{V}(x_i) = \mathbb{P}^{n-1}_{\mathbb{Z}} \subseteq \mathbb{P}^n_{\mathbb{Z}}$ and we are doing an induction proof. The base case of this induction proof is: $\mathbb{P}^0_{\mathbb{Z}} = \text{Proj } \mathbb{Z}[x_0] = \text{Spec } \mathbb{Z}$. The corresponding ring diagram for the above diagram is:



Let $f_{ji} \in K$ be the image of $\frac{x_j}{x_i}$ in K (think about why they do not map to 0). Let $g_i = \nu(f_{i0})$, where $i = 1, \ldots, n$. This is ordered by the valuation, so suppose that g_k is minimal. Now $f_{ik} = f_{ij} f_{jk}$ and $f_{jk} \cdot f_{kj} = 1$ since $\frac{x_i}{x_k} = \frac{x_i}{x_j} \frac{x_j}{x_k}$. Then

$$\nu(f_{ik})s = \nu(f_{ij}) + \nu(f_{jk}) = \nu(f_{ij}) - \nu(f_{kj}) = \nu(f_{i0}) - \nu(f_{k0}) \geq 0$$

Since $\nu(f_{ik}) \geq 0$, the image of $\frac{x_i}{x_j}$ lies in V. Therefore the map $\mathbb{Z}[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}] \longrightarrow V$ exists.
Uniqueness is an exercise. □

6. Quasi-Coherent Sheaves on Schemes

Local	General
Commutative Algebra	Scheme Theory
R	$(X, \mathcal{O}_X)$ scheme
R-module M	$\widetilde{M}$ quasi-coherent sheaf
finitely presented module	coherent sheaf

Remark 6.1. A finitely generated, Noetherian module is finitely presented. A finitely presented module is a finitely generated module with finitely many relations.

Definition 6.2. If $(X, \mathcal{O}_X)$ is a ringed space, an $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ of abelian groups such that for all open $U \subseteq X, \mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module that is compatible with restriction:
For $U' \subseteq U$ open, $m \in \mathcal{F}(U), r \in \mathcal{O}_X(U) \rightsquigarrow r \cdot m \in \mathcal{F}(U)$.
Then $m|_{U'} \in \mathcal{F}(U'), r|_{U'} \in \mathcal{O}_X(U') \rightsquigarrow r \cdot m|_{U'} = r|_{U'} \cdot m|_{U'}$.

Given a ring R and an R-module M, how do I get a sheaf of $\mathcal{O}_X$-modules on the scheme Spec $R = X$?

Let R be a ring, X be Spec R, M be an R-module, and $\widetilde{M}$ be a sheaf of $\mathcal{O}_X$-modules. For all open $U \subseteq X$, we want $\widetilde{M}(U)$ to be an $\mathcal{O}_X(U)$-module. Let $U = D(f) \subseteq \text{Spec } R$. Then

$\mathcal{O}_X(U) = R[\frac{1}{f}]$. Define:

$$\widetilde{M}(D(f)) = M \otimes_R R[\frac{1}{f}] = M[\frac{1}{f}] = \{\frac{m}{f^t} \mid m \in M, t \in \mathbb{Z}\},$$

with the usual equivalence. Since $D(fg) \subseteq D(f)$, we get a map:

$$\widetilde{M}(D(fg)) = M[\frac{1}{fg}] \longleftarrow \widetilde{M}(D(f)) = M[\frac{1}{f}].$$

Now that we have defined $\widetilde{M}$ on all of the basic open sets, we can extend this to all open sets of X as follows:

$$\forall U \subseteq X \text{ open, let } \widetilde{M}(U) = \varprojlim_{D(f)\subseteq U} \widetilde{M}(D(f)) = \varprojlim_{D(f)\subseteq U} M[\frac{1}{f}]$$

This is a module over $\mathcal{O}_X(U) = \varprojlim_{D(f)\subseteq U} \mathcal{O}_X(D(f))$ since each $\widetilde{M}(D(f))$ is a module over $\mathcal{O}_X(D(f))$. Notice that $\widetilde{M}(X) = M$.

Remark 6.3. This is defined in the same way in Shaf.II and in a different, but equivalent, way in Harthorne.

What is the stalk of $\widetilde{M}$ at a point $p \in$ Spec $R = X$?

$$\begin{aligned}
\widetilde{M}_p &= \varinjlim_{p\in U} \widetilde{M}(U) \\
&= \varinjlim_{p\in D(f)} \widetilde{M}(D(f)) \\
&= \varinjlim_{f\notin p} M[\frac{1}{f}] \\
&= M_p \\
&= M \otimes_R R_p
\end{aligned}$$

This is an $\mathcal{O}_{X,p}$-module.

Example 6.4. Let $X = \text{Spec } k[x,y]$, $M_1 = k[x,y] \oplus k[x,y]$, $M_2 = \frac{k[x,y]}{(x,y)}$.
For a point p the stalk:
In M_1 is: $R_p \oplus R_p = (k[x,y] \oplus k[x,y]) \otimes_{k[x,y]} (k[x,y])_p$
In M_2 is: $M_2 \otimes_R R_p$, so this depends on the specific p.

$$M_2 \otimes_R R_p = \frac{k[x,y]}{(x,y)} \otimes_{k[x,y]} k[x,y]_p = \begin{cases} k(p) = k & \text{if } p = (x,y) \\ 0 & \text{otherwise} \end{cases}$$

$\widetilde{M_2}$ is the skyscraper sheaf at $p = (x,y)$. Spec $k(p) \xrightarrow{i} \mathbb{A}^2_k$, where $\widetilde{M_2} = i_*\widetilde{k(p)}$.

Lecture 12. February 17, 2009

Exercise 6.5. Hartshorne: 5.1, 5.4, 5.6, 5.7, exercises 1-5 from Karen's "pullbacks and all that" handout, and as one exercise:

- **(a):** Look up the definition and basics of injective modules in an algebra text (eg Dummit and Foote).
- **(b):** Prove that every module over a commutative ring admits an injective resolution.
- **(c):** If S is an R algebra and E is an injective R-module, prove that $HOM_R(S,E)$ is an injective S-module (hint: adjointness of tensor and hom).

(Quasi-)Coherent Sheaves on Schemes:

Definition 6.6. M module over a commutative ring A, the **quasi-coherent** sheaf of $\mathcal{O}_X$-modules is defined by: $\widetilde{M}(D(g)) = M[\frac{1}{g}] \equiv M_g = M \otimes_A A[\frac{1}{g}]$ where

$$\widetilde{M}(D(g)) = \{\frac{m}{g^t} \mid m \in M, t \in \mathbb{Z} \text{ where } \frac{m_1}{g^{t_1}} \sim \frac{m_2}{g^{t_2}} \Leftrightarrow \exists s \text{ such that } g^s(m_1 g^{t_2} - m_2 g^{t_1}) = 0\}$$

Let $U = \cup D(g_i)$. Then $m \in \widetilde{M}(U) \Leftrightarrow m_i \in \widetilde{M}(D(g_i)), \forall i$ and the m_i agree on overlaps. So that $\widetilde{M}(U) = \varprojlim_{D(g) \subseteq U} \widetilde{M}(D(g))$.

Definition 6.7. $\widetilde{M}$ is a **coherent** sheaf if it is a quasi-coherent sheaf and M is a finitely presented module.

Definition 6.8. A **quasi-coherent sheaf** on a scheme $(X, \mathcal{O}_X)$ is a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules such that X admits an affine cover $X = \cup U_i$ (each $U_i = \text{Spec } A_i$) with $\mathcal{F}|_{U_i} = \widetilde{M}_i$ for some A_i-module M_i. A quasi-coherent sheaf $\mathcal{F}$ is **coherent** if each M_i is finitely presented.

Examples:

- $\mathcal{O}_X$ is coherent on any scheme
- $\mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X$ is quasi-coherent (there are any number of copies of the $\mathcal{O}_X$)
- $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module (i.e. $X = \cup U_i$ be an open cover of X such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i} \oplus \cdots \oplus \mathcal{O}_{U_i} \forall i$, where $\mathcal{O}_{U_i} \equiv \mathcal{O}_X|_{U_i}$.
- X smooth, irreducible variety over k and D a divisor, $\mathcal{L}(D) = \{\varphi \mid div\varphi + D \geq 0\} \subset k(X)$ is coherent
- $\wedge^p \Omega_{X/k}$ modules of differentials, where X is smooth, irreducible variety, is also coherent
- $Y \subseteq^i X, \mathcal{O}_X \longrightarrow i_* \mathcal{O}_Y$ kernel is sheaf of ideals $\mathcal{I}_Y \subseteq \mathcal{O}_X$. Then $\mathcal{I}_Y$ is quasi-coherent and if $\mathcal{O}_X$ is Noetherian, then so is $\mathcal{I}_Y$.

Theorem 6.9. *(Prop 5.2 in Hartshorne) Let $X =$ Spec A. The functor $\{A\text{−}mod\} \longrightarrow \{\mathcal{O}_X\text{−}mod\}$ sending $M \longrightarrow \widetilde{M}$ is an exact, fully faithful functor, whose image is the category of quasi-coherent $\mathcal{O}_X$-modules.*

Proof. Given an exact sequence of A-modules:

$$0 \to M_1 \to M_2 \to M_3 \to 0 \Rightarrow^? 0 \to \widetilde{M_1} \to \widetilde{M_2} \to \widetilde{M_3} \to 0 \text{ exact}$$

This is true $\Leftrightarrow \forall p \in$ Spec A, the induced sequence of $\mathcal{O}_{X,p}$-modules is exact on stalks.

$$0 \to (\widetilde{M_1})_p = (M_1)_p \to (\widetilde{M_2})_p = (M_2)_p \to (\widetilde{M_3})_p = (M_3)_p \to 0$$

$(M_1)_p = M_1 \otimes_A A_p$. From commutative algebra fact is that a short sequence of modules is exact if and only if its localization is exact for all localizations. Given a map of A-modules, $M \xrightarrow{f}$, we get an induce map of $\mathcal{O}_X$-modules, $\widetilde{M} \xrightarrow{\tilde{f}} \widetilde{N}$. Fully faithful means that every map $\widetilde{M} \xrightarrow{g} \widetilde{N}$ of $\mathcal{O}_X$-modules is induced by some map $M \longrightarrow N$ of A-modules (via $\sim$). Let $M = \widetilde{M}(x)$ and $N = \widetilde{N}(X)$. Then $M = \widetilde{M}(X) \xrightarrow{g(X)} \widetilde{N}(X) = N$ is an $\mathcal{O}_X(X) = A$-module map.

$$Hom_{A-mod}(M, N) \xrightarrow[\text{bijection}]{\sim} Hom_{\mathcal{O}_X-mod}(\widetilde{M}, \widetilde{N})$$

□

Corollary 6.10. *The category of quasi-coherent modules on an affine scheme Spec A is equivalent to the category of A-modules.*

Proposition 6.11. *If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module on a scheme $(X, \mathcal{O}_X)$, then for every affine open U, we have $\mathcal{F}|_U = \widetilde{M}$, for some M an $\mathcal{O}_X(U)$-module. (set $M = \mathcal{F}(U)$).*

Proof. If $\mathcal{F}$ is quasi-coherent, there is a basis $\mathcal{B}$ for X such that $\forall U \in \mathcal{B}$, $\mathcal{F}|_{U_\lambda} m_\lambda$ for some A_λ-module M_λ on $U_\lambda = \text{Spec } A_\lambda$. $\{U_i\}$, where $U_i = \text{Spec } A_i$, for some affine cover $\mathcal{F}|_{U_i} = M_i$, M_i is an A_i-module. By covering U_i's by basic open affines, we get $\mathcal{B}$. □

All basic commutative algebra constructions for A-modules can be done for quasi-coherent sheafs on a scheme.

Example 6.12. $\mathcal{F} \xrightarrow{f} \mathcal{G}$ is a morphism of quai-coherent $\mathcal{O}_X$-modules. What sheafs do we get out of this?

- $\text{Ker}(f)$ is a quasi-coherent sheaf (since localization commutes with taking the kernel)
- $\text{Im}(f)$ (we must look at sufficiently small open sets for this to be a sheaf)
- $\text{Coker}(f)$

Remark 6.13. Given $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of quasi-coherent sheaves on X. For $U \subseteq X$, we get a sequence $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$ that may not be exact. This is exact if U is affine or if it is contained in an affine open set, but it might not be exact otherwise.

6.0.2. *Push Down (Direct Image) of Quasi-Coherent Modules.*
Let $X \xrightarrow{f} Y$ be a morphism of schemes and $\mathcal{F}$ be a quasi-coherent scheme.
$f_*\mathcal{F}$ is the sheaf on Y given by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$. This is an $\mathcal{O}_Y$-module and

$$\mathcal{O}_Y(U) \xrightarrow[\text{of scalars}]{\text{restriction}} f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U)).$$

If $\mathcal{F}$ is a quasi-coherent sheaf on X and either X is noetherian or f is separated, then $f_*\mathcal{F}$ is quasi-coherent.

Definition 6.14. A morphism $X \xrightarrow{f} Y$ of schemes is an **affine map** if the pre-image of every affine set is affine. Equivalently, the pre-image of an affine cover is affine.

Suppose $\mathcal{F}$ is quasi-coherent. Cover X by affine U_i such that $\mathcal{F}|_{U_i} = \widetilde{M_i}$ ($M_i = \mathcal{F}(U_i)$ considered as an $\mathcal{O}_X(U_i) = A_i$-module). This is easy when f is an affine map since we can cover Y by $U_i = \text{Spec } A_i$, so that $f_*\mathcal{F}|_{U_i} = \mathcal{F}(f^{-1}(U_i))$ and $f^{-1}(U_i)$ is affine.

Prove that if $\mathcal{F}$ is quasi-coherent, then $f_*\mathcal{F}$ is quasi-coherent on Y. Assume that $X \xrightarrow[\text{separated}]{f} Y$ and, without loss of generality, Y is affine.

$$s \in f_*\mathcal{F}(U) \Leftrightarrow s_i \in \mathcal{F}(U_i)\forall i \text{ and the } s_i \text{ agree on overlaps.}$$

$$0 \to f_*\mathcal{F} \to \oplus_i f_*\mathcal{F}|_{U_i} \to \oplus_{i,j} f_*\mathcal{F}|_{U_i \cap U_j} \text{ is exact}$$

Push Down preserves the category of quasi-coherent modules.

Example 6.15 (Of Push Downs). Let $A \xrightarrow{\varphi} B$ and $X = \text{Spec } B \xrightarrow{f} \text{Spec } A = Y$ be induced by φ. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then $\mathcal{F} = \widetilde{M}$ for some B-module M. $f_*\mathcal{F}$ is a quasi-coherent on Y, $f_*\mathcal{F} = M$ (now thinking of M as an A-module via retriction of scalars $A \to B$). If $\mathcal{F}$ is coherent, A and B Noetherian, is $f_*\mathcal{F}$ coherent? Not in general. For instance:
Let $k \hookrightarrow k[t], A^1_k \xrightarrow{f} \text{Spec } k, \mathcal{F} = \mathcal{O}_{\mathbb{A}^1} = \widetilde{k[t]}$ so that $f_*\mathcal{F}(\text{Spec } k) = \mathcal{F}(f^{-1}(\text{Spec } k)) = \mathcal{F}(\mathbb{A}^1_k) = k[t]$ as a k-module, which is definitely not finitely generated as a k-module.
If f is a finite map, then $f_*\mathcal{F}$ is coherent.

Lecture 13. February 19, 2009

Given any map of topological spaces $X \xrightarrow{f} Y$ and $\mathcal{F}$ a sheaf on X that is an $\mathcal{O}_X$-module, we get a sheaf on Y: $f_*\mathcal{F}$ and a map of ringed spaes $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$. In the case where f is a morphism of schemes and $\mathcal{F}$ is a a quasi-coherent $\mathcal{O}_X$-module, then $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module. However if $\mathcal{F}$ is coherent, $f_*\mathcal{F}$ need not be coherent.

6.0.3. *Pullbacks:*

Commutative Algebra	Ringed Spaces
$A \longrightarrow B$	$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$
$\{B\text{-mod}\} \xrightarrow[\text{of scalars}]{\text{restriction}} \{A\text{-mod}\}$	$\{\mathcal{O}_X\text{-mod}\} \xrightarrow{f_*} \{\mathcal{O}_Y\text{-mod}\}$
$\{B\text{-mod}\} \longleftarrow \{A\text{-mod}\}$	$\{\mathcal{O}_X\text{-mod}\} \longleftarrow \{\mathcal{O}_Y\text{-mod}\}$
where $B \otimes_A B \longleftarrow N$	$f^*\mathcal{F} \longleftarrow \mathcal{F}$

Generalities:
Given $X \xrightarrow{f} Y$ a map in the category of topological spaces between such spaces.
Let $\mathcal{F}$ be a sheaf on Y and we want to define $f^{-1}\mathcal{F}$ a sheaf on X.
If f is an open map we can define a presheaf $f^{-1}\mathcal{F}(U) = \mathcal{F}(f(U))$, where $U \subset X$ is open, which defines a unique sheaf $f^{-1}\mathcal{F}$.
Otherwise we can define $f^{-1}\mathcal{F}$ by looking at all open sets that contain $f(U)$ for a given open $U \subset X$.

Definition 6.16. $f^{-1}(\mathcal{F})$ is the sheaf associated to the presheaf

$$f^{-1}(\mathcal{F})(U) = \varinjlim_{V \supseteq f(U) \text{ open in } Y} \mathcal{F}(V)$$

This definition agrees with the open we had when f was an open map since in that case we have a terminal object in the direct limit, which is $f(U)$.

Example 6.17. Let $U \subseteq_j^{\text{open}} X$ and $\mathcal{F}$ a sheaf on X. Let $V \subset U$ open. Then:

$$j^{-1}\mathcal{F}(V) = \mathcal{F}(j(V)) = \mathcal{F}(V) \text{ note that } \mathcal{F}|_U = j^{-1}\mathcal{F}$$

Example 6.18. Let $\mathbb{C} \overset{i}{\hookrightarrow} \mathbb{C}^2$ be given by $z \mapsto (z, 0)$ and $U \subseteq \mathbb{C}$ open. Then

$$i^{-1}\mathcal{O}^{an}_{\mathbb{C}^2}(U) = \varinjlim_{V \supseteq i(U) \text{ open}} \mathcal{O}^{an}_{\mathbb{C}^2}(V) = \text{ germs of analytic functions defined on open neighborhood of } i(U)$$

Example 6.19. Let $\{p, q\} \overset{i}{\hookrightarrow} \mathbb{A}^1_{\mathbb{C}}$ and $\underline{\mathbb{C}}$ be the constant sheaf on $\mathbb{A}^1_{\mathbb{C}}$.

$$i^{-1}\underline{\mathbb{C}}(\{p, q\}) = \varinjlim_{V \supseteq \{p,q\}} \underline{\mathbb{C}}(V) = \mathbb{C}$$

$$i^{-1}\underline{\mathbb{C}}(\{p\}) = \varinjlim_{V \supseteq \{p\}} \underline{\mathbb{C}}(V) = \mathbb{C}$$

$$i^{-1}\underline{\mathbb{C}}(\{q\}) = \varinjlim_{V \supseteq \{q\}} \underline{\mathbb{C}}(V) = \mathbb{C}$$

So this is only a presheaf.

Example 6.20. $k[X] \hookrightarrow k[x, y]$, then $\mathbb{A}^1_k = \text{Spec } k[x] \xleftarrow{\pi} \text{Spec } k[x, y] = \mathbb{A}^2_k$. Note that π is an open map.

$$\pi^{-1}\mathcal{O}_{\mathbb{A}^1_k}(D(x - \lambda)) = \varinjlim_{V \supseteq \pi(D(x-\lambda))} \mathcal{O}_{\mathbb{A}^1_k}(V) = \mathcal{O}_{\mathbb{A}^1_k}(D(x - \lambda)) = k[x]\left[\frac{1}{x - \lambda}\right]$$

Let $f \in k[x, y]$. If the image of f in $\mathbb{A}^2_k$ projected onto $\mathbb{A}^1_k$, then $\pi^{-1}\mathcal{O}_{A^1_k}(D(f)) = k[x]$. So $\pi^{-1}\mathcal{O}_{\mathbb{A}^1_k}$ is a sheaf of rings on $\mathbb{A}^2$, but not $\mathcal{O}_{\mathbb{A}^2_k}$.

Let $(X, \mathcal{O}_X) \xrightarrow{(f,f_*)} (Y, \mathcal{O}_Y)$ be a map of ringed spaces (so that $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$). **Note:** $f^{-1}\mathcal{O}_Y$ is a sheaf of rings on X (but not usually $\mathcal{O}_X$ as we saw in the previous example). There is a natural map of sheaves of rings on X:

$$f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$$

For open $U \subseteq X$: $f^{-1}\mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(U)$, where

$$\varinjlim_{V \supseteq f(U) \text{ open}} f^{-1}\mathcal{O}_Y(U) \xrightarrow[\text{ringed spaces}]{\text{induced by map of}} \varinjlim_{V \supseteq f(U)} \mathcal{O}_X(f^{-1}(V)) \xrightarrow{\text{restriction}} \mathcal{O}_X(U)$$

Given an $\mathcal{O}_Y$-module (i.e. a sheaf of $\mathcal{O}_Y$-modules), $\mathcal{F}$, $f^{-1}\mathcal{F}$ is an $f^{-1}\mathcal{O}_Y$-module. $f^{-1}\mathcal{F}$ is an $\mathcal{O}_Y$-module as follows: for $U \subseteq X$ open, $r \in f^{-1}\mathcal{O}_Y(U) = \varinjlim \mathcal{O}_Y(V)$ and $m \in f^{-1}\mathcal{F}(U) = \varinjlim \mathcal{F}(V) \Rightarrow rm \in f^{-1}\mathcal{F}(V)$

Definition 6.21. Given an $\mathcal{O}_Y$-module $\mathcal{F}$, $f^*\mathcal{F}$ is the sehaf of $\mathcal{O}_X$-modules on X associated to the presheaf given by for $U \subseteq X$ open, $f^*\mathcal{F}(U) = \mathcal{O}_X(U) \otimes_{f^{-1}\mathcal{O}_Y(U)} f^{-1}\mathcal{F}(U)$.

Example 6.22. If $X \xrightarrow{f} Y$, then $f^*\mathcal{O}_Y = \mathcal{O}_X$.

Example 6.23. Let $A \xrightarrow{\varphi} B$ and Spec $B \xrightarrow{f}$ Spec A. We get $f^*\widetilde{M} = B \otimes_A M = \widetilde{M}$

Proposition 6.24. *If $\mathcal{F}$ is a locally free $\mathcal{O}_Y$-module, then $f^*\mathcal{F}$ is a locally free $\mathcal{O}_X$-module.*

Proposition 6.25. *If $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, then*

$$\{\mathcal{O}_Y - \textit{modules}\} \xrightarrow{f^*} \{\mathcal{O}_X - \textit{modules}\} \textit{ is a functor: } \mathcal{O}_Y \longrightarrow \mathcal{O}_X = f^*\mathcal{O}_Y$$

If f happens to be a morphism of schemes, then the quasi-coherent (coherent) modules on $\mathcal{O}_Y$ pullback to quasi-coherent (coherent) modules on $\mathcal{O}_X$.

Example 6.26 (Of Quasi-Coherent Sheaves on Proj$S$). Let $S = S_0 \oplus S_1 \oplus \dots$ and $S_{>0}$ be the irrelevant ideal. This has a basis $D_+(f) = \text{Spec } \left[S\left[\frac{1}{f}\right]\right]_0$, where f is homogeneous. Take M a $\mathbb{Z}$-graded S-module. This determines a quasi-coherent sheaf $\widetilde{M}$ on $X = \text{Proj } S$. On basic open set $D_+(f)$, $\widetilde{M}(D_+(f)) = \left[M\left[\frac{1}{f}\right]\right]_0$ is a module over $\mathcal{O}_X(D_+(f)) = \left[S\left[\frac{1}{f}\right]\right]_0$.
Check that this is quasi-coherent: If $D_+(fg) \subseteq D_+(f)$ (whenever we have open basic open set inside another, we can assume it has this form and in fact we can assume that f and g are homogeneous of the same degree by looking at higher powers of f and g). We need:

$$\widetilde{M}(D_+(fg)) = \widetilde{\widetilde{M}(D_+(f))}\left(\frac{1}{\frac{g}{f}}\right)$$

$$\text{Note that } \left[M\left[\frac{1}{fg}\right]\right]_0 = \widetilde{M}(D_+(fg)) =^? \left[M\left[\frac{1}{f}\right]_0\right]\left[\frac{f}{g}\right]$$

Lecture 14. March 2, 2009

Definition 6.27. A **quasi-coherent sheaf** $\mathcal{F}$ on a scheme X is an $\mathcal{O}_X$-module with the property that for every open affine $U \subseteq X$ (say $U = \text{Spec } A$) $\mathcal{F}|_U = \widetilde{M}$ where M is the $\mathcal{O}_X(U) = A$-module $\mathcal{F}(U)$.

7. Quasi-Coherent Sheaves on Projective Schemes over A

Setup:
Let $S = S_0 \oplus S_1 \oplus S_2 \oplus \dots$ be an $\mathbb{N}$-graded ring, where $S_0 = A$ is a Noetherian ring and $S_{>0}$ is the irrelevant ideal. Let $X = \text{Proj } S$, so we have a map $X \longrightarrow \text{Spec } A$. The closed sets of X are $\mathbb{V}(I)$, where I is a homogeneous ideal, and the basic open sets are $D_+(f) = \{p \in \text{Proj } S \mid f \notin p\}$. $\mathcal{O}_X(D_+(f)) = \left(S\left[\frac{1}{f}\right]\right)_0$.

Definition 7.1. Let M be a $\mathbb{Z}$-graded S-module. The associated sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ is defined as: $\widetilde{M}(D_+(f)) = \left[M\left[\frac{1}{f}\right]\right]_0$, where f is homogeneous.

Note:
This is an $\mathcal{O}_X(D_+(f)) = \left[S\left[\frac{1}{f}\right]\right]_0$-module.
If $D_+(g) \subseteq D_+(g)$, we need a natural restriction map (we can assume that $g = fh$ and $\deg f = \deg h$).

$$\widetilde{M}(D_+(f)) = \left[S\left[\frac{1}{f}\right]\right]_0 \longrightarrow \widetilde{M}(D_+(g)) = \left[S\left[\frac{1}{fh}\right]\right]_0$$

$$\text{by } \frac{m}{f^t} \mapsto \frac{mh^t}{(fh)^t}$$

Proposition 7.2. *This $\mathcal{O}_X$-module $\widetilde{M}$ is quasi-coherent.*

Proof. $\left[S\left[\frac{1}{fh}\right]\right]_0 = \left[S\left[\frac{1}{f}\right]\right]_0[(\frac{h}{f})^{-1}]$, where $\frac{h}{f} \in \left[S\left[\frac{1}{f}\right]\right]_0$. □

Main Fact: If S is finitely generated as an A-algebra by its element of degree 1, then all quasi-coherent sheaves on Proj S are of this form and there is a functor:

$$\{\mathbb{Z}\text{-graded, finitely generated } S\text{-modules}\} \longrightarrow \{\text{coherent } \mathcal{O}_X\text{-modules}\}$$

This is not an equivalence of categories.

Example 7.3. Take any graded S-module, M, and let $M' \subseteq M$ be any submodule such that $\exists N_0 \in \mathbb{Z}$, with $M'_N = M_N \forall N \geq N_0$. The natural map $\widetilde{M'} \longrightarrow \widetilde{M}$ of $\mathcal{O}_X$-modules is an isomorphism. For instance:

$$\widetilde{M'}(D_+(f)) \longrightarrow \widetilde{M}(D_+(f)) \text{ given by } \frac{m}{f^t} \leftrightarrow \frac{mf^{N_0}}{f^{t+N_0}}$$

So for the above functor, $M \sim M'$ if they agree in large degree ($\exists N_0$ such that $M_N = M'_N \forall N \geq N_0$).

7.0.4. *Twisting Sheaves on Proj S.*

Definition 7.4. Given a $\mathbb{Z}$-graded module M over S, **M shifted by** $N \in \mathbb{Z}$ is the graded S-module $M(N)$, which is M as an S-module but the grading is shifted: $\left[M[N]\right]_d = M_{N+d}$.

Caution: The different $M(N)$ are almost always different graded S-modules

Example 7.5. Fix $S = A[x_0, \dots, x_n]$. Let $M = S$ and lets study $\widetilde{S(N)} = \mathcal{O}_X(N)$ (the N^{th} twist of the structure sheaf $\mathcal{O}_X$ on $X = \text{Proj } S$).

$$\begin{aligned}
\mathcal{O}_X(N)(D_+(x_i)) &= \left[S(N)\left[\frac{1}{x_i}\right]\right]_0 \\
&= \left[S\left[\frac{1}{x_i}\right]\right]_N \\
&= \{\frac{s}{x_i^t} \mid \deg s = N+t, s \text{ is a sum of monomials} s = x_0^{a_0}\dots x_n^{a_n}, \sum a_i = N+t\} \\
&= \{\frac{x_0^{a_0}}{x_i^{a_0}} \cdots \frac{x_n^{a_n}}{x_i^{a_n}}\} \\
&= A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] x_i^N
\end{aligned}$$

This is not the same as S even though they are isomorphic. Also, in this example we did not use the assumption that there were no relations on the x_i.

Proposition 7.6. *If S is finitely generated as an A-algebra by its element of degree 1, the twisting sheaves $\mathcal{O}_X(n)$ on $X = Proj\ S$ are all locally free $\mathcal{O}_X$-modules of rank 1.*

Compute the global sections of $\mathcal{O}_X(n)$ in the case that $X = \text{Proj } A[x_0, \dots, x_N]$. Take any $s \in \mathcal{O}_X(n)(X) = \Gamma(X, \mathcal{O}_X(n))$.

$$s|_{D_+(x_i)} = x_i^n \frac{F_i(x_0, \dots, x_N)}{x_i^t},$$

where F_i is homogeneous of degree t and it depends on the patch $D_+(x_i)$. When we look at another patch $D_+(x_j)$, we get a different homogenous polynomial F_j. On the overlap, $D_+(x_i) \cap D_+(x_)$, we want $\frac{x_i^n F_i}{x_i^{t_i}} = \frac{x_j^n F_j}{x_j^{t_j}}$. We can assume that $t_i = t_j$ and $\deg(F_i) = \deg(F_j)$ since if $t_i < t_j$ then we could replace F_i by $F_i x_i^{t_j - t_i}$.

Check cases:

- If $n < 0$, this is impossible for F_i, F_j unless they are both zero.
- If $n = 0$ (this is the case of $\mathcal{O}_X$, the global sections are A.
- If $n > 0$: $\Gamma(X, \mathcal{O}_X(n)) = S_n$.

  For example, when $n = 1$: $\left[S[\frac{1}{x_i}]\right]_1 \ni x_i \frac{x_j}{x_i} = x_j \in \left[S[\frac{1}{x_j}]\right]_1$

There is always a map for a graded S-module M:

$$[M]_0 \longrightarrow \Gamma(X, \widetilde{M}) \text{ where}$$

$$m \mapsto m \text{ which on } D_+(x_i) \text{ is } \frac{m}{1} \in \left[M\left[\frac{1}{x_i}\right]\right]_0$$

Observe that $S = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(n))$.

Definition 7.7. Given a quasi-coherent sheaf $\mathcal{F}$ on $X = \text{Proj } S$, the n-th twist of $\mathcal{F}$ is the quasi-coherent sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

If S is finitely generated as an A-algebra by its element of degree 1, $\widetilde{M}(n) = \widetilde{M(n)}$.
There is a functor:

$$\{\mathbb{Z}\text{-graded, finitely generated} S\text{-modules}\}/\sim \longrightarrow \{\text{coherent } \mathcal{O}_X\text{-modules}\}$$

$$M \longleftarrow \widetilde{M}$$

$$M = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n)) \equiv \Gamma^*(\mathcal{F}) \longleftarrow \mathcal{F}$$

We need to check that $\bigoplus_{n\in\mathbb{Z}}\Gamma(X,\mathcal{F}(n))$ is a graded S-module:

$$s \in S_d = \Gamma(X,\mathcal{O}_X(d)) \text{ and } m \in M_t = \Gamma(X,\mathcal{F}(t)) = \Gamma(X,\mathcal{F}\otimes\mathcal{O}_X(t))$$
$$s\cdot m \in \Gamma(X,\mathcal{F}\otimes\mathcal{O}_X(t+d)) = M_{d+t}$$

Again assuming that S is finitely generated as an A-algebra by its element of degree 1,

$$\mathcal{O}_X(n)\otimes_{\mathcal{O}_X}\mathcal{O}_X(m) \xrightarrow{\cong} \mathcal{O}_X(n+m)$$

Theorem 7.8. *Under the assumption that S is finitely generated as an A-algebra by its element of degree 1, this functor defines an equivalence of categories.*

Lecture 15. March 5, 2009

Let M be a graded S-module (S-graded ring) and $p \in \text{Proj } S$. Let U = homogeneous elements in $S - p$.

$$\widetilde{M_p} = \Big[M\big[U^{-1}\big]\Big]_0 = M_{<p>}$$

Theorem 7.9. *Fix $S = A[x_0,\dots,x_n]/I$ a graded ring (with $\deg x_i = 1$). Let X = Proj S.*

$$\{\textit{ Finitely graded } S\textit{-modules}\}/\sim \longrightarrow \{\textit{coherent sheaves on } X\}$$
$$M \longrightarrow \tilde{M}$$
$$\Gamma_*\mathcal{F} = \oplus_n\Gamma(X,\mathcal{F}(n)) \longleftarrow \mathcal{F}$$

where $M\sim M'$ if they agree in high degree, $\mathcal{O}_X(n) = \widetilde{S(n)}$, $\mathcal{F}(n) = \mathcal{F}\otimes\mathcal{O}_X(n)$, and S is the torsion free cyclic S-module generated by a degree $-n$ element.

Sketch. Need to show that:

- $\widetilde{\Gamma_*(\mathcal{F})} \longrightarrow \mathcal{F}$ there is a natural map which is an isomorphism (Hart.: Prop. 5.15). We can describe this map on a basic open set $D_+(f)$, where f is homogeneous.
$$\widetilde{\Gamma_*(\mathcal{F})}(D_+(f)) \longrightarrow \mathcal{F}(D_+(f))$$
Let m be a degree n element of $\Gamma_*(\mathcal{F})$ for $n = t\deg f$ (i.e. $m \in \Gamma(X,\mathcal{F}(n))$). We define this map to be:
$$\frac{m}{f^t} \mapsto m|_{D+(f)}\cdot\frac{1}{f^t}$$
Note: $\frac{1}{f^t} \in [S[\frac{1}{f}]]_{-n} = [S(-n)(\frac{1}{f}]_0$, $m \in \mathcal{F}\otimes\mathcal{O}_X(n)(D_+(f))$, $\frac{1}{f} \in \mathcal{O}_X(-n)(D_+(f))$.
Check that this map works (for instance that it commutes with restriction).
- $M \longrightarrow \Gamma_*(\tilde{M})$ there is a natural map which is an isomorphism in high degree (i.e. a morphism of graded S-modules and for large $N_0 \in \mathbb{Z}$ such that $\forall N \geq N_0$ this is a bijection: $M_N \longrightarrow (\Gamma_*(\tilde{M}))_0$).
Defining this map: Take $m \in M$ of degree n.
$$m \mapsto m \in \Gamma(X,\tilde{M}(n)), \text{ where } \tilde{M}(n) = \tilde{M}\otimes_{\mathcal{O}_X}\mathcal{O}_X(n) = \widetilde{M(n)}$$
Viewing m as an element of $M(n)$, then m has degree 0. m is a global section since it is a degree 0 element.
$$\frac{m}{1} \in \widetilde{M(n)}(D_+(f)) = \left[M\left[\frac{1}{f}\right]\right]_0 = \left[M\left[\frac{1}{f}\right]\right]_n$$
- Given a map $M \xrightarrow{\varphi} N$, show there is a natural map $\tilde{M} \longrightarrow \tilde{N}$ that preserves the grading. In particular we have a map:
$$\tilde{M}(D_+(f)) = [M[\frac{1}{f}]]_0 \longrightarrow \tilde{N}(D_+(f)) = [N(\frac{1}{f}]]_0 \text{ where } \frac{m}{f^t} \mapsto \frac{\varphi(n)}{f^t}.$$

□

Remark 7.10. The category of graded S-modules is not the same as the category of modules over the graded ring S.

Definition 7.11. A sheaf $\mathcal{F}$ of (quasi-coherent) $\mathcal{O}_X$-modules on a scheme X is **globally generated** if $\exists\{S_i\}_{i\in I} \in \Gamma(X,\mathcal{F})$ such that the restriction of the S_i to every affine open set U generates $\mathcal{F}(U)$ as an $\mathcal{O}_X(U)$-module.

Equivalently, the germs of the $\{S_i\}$ generate $\mathcal{F}_p \forall p$ (as an $\mathcal{O}_{X,p}$-module).
Equivalently, $\mathcal{F}$ is a quotient of a free $\mathcal{O}_X$-module: $\oplus_{i\in I}\mathcal{O}_X \twoheadrightarrow \mathcal{F}$ (let $e_i = (0,\dots,0,1,0,\dots,0)$ where the 1 is in the i^{th} coordinate). Then this map is given by: $e_i \mapsto S_i$.

Examples:

- Free $\mathcal{O}_X$-modules, $\oplus\mathcal{O}_X$ (in particular $\mathcal{O}_X$)
- Any quasi-coherent sheaf on affine scheme
- On $X = \text{Proj } S$, $\mathcal{O}_X(N), N < 0$ is NOT globally generated
- On $X = \text{Proj } S$, $\mathcal{O}_X(N), N > 0$ then $\Gamma(X,\mathcal{O}_X(N)) \supseteq S_N \ni m$ and $m \in [S(N)]_0 \longleftarrow m$
- Let M be a graded S-module generated in degree 0, $X = \text{Proj } S$, and $\mathcal{F} = \tilde{M}$.

$$[M]_0 \subseteq \mathcal{F}(X)$$
$$\mathcal{F}_p = M_{<p>}$$
$$\{m_i\}_{i\in I} \text{ degree 0, generated for } M.$$
$$\mathcal{F}(D_+(f)) = \left[M[\left[\frac{1}{f}\right]\right]_0$$

Take an element $m \in M$ of degree $t \deg f$. Let $\{s_i\}$ all have degree 0. Then:

$$\frac{m}{f^t} = \frac{s_1 m_1 + \cdots + s_k m_k}{f^t} = \frac{s_1}{f^t}\frac{m_1}{1} + \cdots + \frac{s_k}{f^t}\frac{m_k}{1}$$

So that $\frac{m_i}{1} \in [M[\frac{1}{f}]]_0$.

Fujita's Freeness Conjecture
Let X be a smooth projective variety over $k = \overline{k}$ so that $\omega_X = \wedge^d\Omega_X$ is a coherent sheaf on X.
Conjecture: $\omega_X(N), N \geq \dim X + 1$ is globally generated.

Part 3. Introduction to Cohomology

Example 7.12. Ext^p: Fix a ring, R and an R-module A. Consider the functor $\text{Hom}_R(A,\cdot)$ which sends R-modules to R-modules.

$$\{R-\text{modules}\} \longrightarrow \{R-\text{modules}\}$$
$$M \longrightarrow \text{Hom}_R(A,M)$$
$$(M \xrightarrow{f} N) \longrightarrow (\text{Hom}_R(A,M) \longrightarrow \text{Hom}_R(A,N))$$

This is a covariant and left exact functor, i.e. given an exact sequence:

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \longrightarrow M_3 \longrightarrow 0, \text{ we get the exact sequence:}$$
$$0 \longrightarrow \text{Hom}(A,M_1) \longrightarrow \text{Hom}(A,M_2) \longrightarrow \text{Hom}(A,M_3)$$

For every covariant left exact functor, there are **right derived functors** called $\text{Ext}^p(A,\cdot)$. Defined as follows:
To compute $\text{Ext}^p(A,M)$

(1) Take an injective resolution of M: $0 \longrightarrow M \hookrightarrow I^1 \longrightarrow I^2 \longrightarrow$. Let $I^\cdot$ be the complex $0 \longrightarrow I^0 \hookrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots$

(2) Apply the functor to the complex $I^{\cdot}$:

$$0 \longrightarrow \mathrm{Hom}_R(A, I^0) \longrightarrow \mathrm{Hom}_R(A, I^1) \longrightarrow \mathrm{Hom}_R(A, I^3) \longrightarrow \dots$$

(3) By definition, Ext^p =the p^{th} cohomology= $\frac{ker}{\mathrm{Im}}$ at the p^{th} position.

Definition 7.13. I is **injective** means that $\mathrm{Hom}(\cdot, I)$ is exact.

Lecture 16. March 10, 2009

Exercise 7.14. Read Hartshorne III: §1,2,3(,4). Exercises in III: 2.2,2.4,2.7,3.1,3.2.

Review of Basic Homological Algebra
A **co-chain complex** is objects in an abelian category, A_i, with maps between these objects:

$$\dots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \longrightarrow \dots,$$

where $d^2 = 0$ (equivalently, $\mathrm{Im}d^{i-1} \subseteq \mathrm{Ker}d^i \subseteq A^i$). Whenever we have a co-chain complex we can compute its cohomology:

$$H^i(A^{\cdot}) = \frac{\mathrm{Ker}d^i}{\mathrm{Im}d^{i-1}}.$$

This sequence is exact if $\mathrm{Im}d^{i-1} = \mathrm{Ker}d^i, \forall i$ (equivalently, the cohomology is zero for all i). Let C, C' be abelian categories. In particular, we consider the following abelian categories:

- Ab
- R-mod
- Sheaves of abelian groups on a topological space X
- Quasi-coherent sheaves on a scheme
- Co-chain complexes with morphisms between co-chains

Given a left exact covariant functor $C \xrightarrow{\Gamma} C'$, you can always (*provided your category has enough injectives*) compute its **right derived functors** $R^p\Gamma : C \longrightarrow C'$ ("the p^{th} Right derived functor of Γ") with the following properties:

(1) $R^0\Gamma(\mathcal{F}) = \Gamma(\mathcal{F}), \forall \mathcal{F} \in Ob(C)$
(2) Given any short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in C, there exists a long exact sequence

$$0 \longrightarrow R^0\Gamma(A) \longrightarrow R^0\Gamma(B) \longrightarrow R^0\Gamma(C) \longrightarrow R^1\Gamma(A) \longrightarrow R^1\Gamma(B) \longrightarrow R^1\Gamma(C) \longrightarrow \cdots.$$

(3) The $R^p\Gamma$ are "universal" with respect to the previous two conditions.

Remark 7.15. $R^p\Gamma$ does not depend on the choice of injective resolutions.

Example 7.16. $C = C' = R$-module, $\Gamma = \mathrm{Hom}_R(A, \cdot)$ and $A \xrightarrow{g} M \xrightarrow{f} N$, then $\mathrm{Hom}_R(A, M) \longrightarrow \mathrm{Hom}_R(A, N)$. Since Γ is a left exact functor, given an exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow Q \longrightarrow 0,$$

we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(A, m) \longrightarrow \mathrm{Hom}_R(A, N) \longrightarrow \mathrm{Hom}_R(A, Q).$$

Ext is the right derived functor of $\mathrm{Hom}_R(A, \cdot)$ (i.e. $R^p\mathrm{Hom}(A, \cdot) = Ext^p_R(A, \cdot)$), so that we get the long exact sequence:

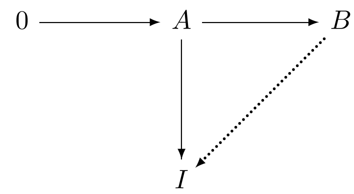
$$Ext^1(A, M) \longrightarrow Ext^1(A, N) \longrightarrow Ext^1(A, Q) \longrightarrow Ext^2(A, M) \longrightarrow \dots.$$

Example 7.17. Let C be the category of sheaves of abelian groups on topological space X.

$$C \xrightarrow{\Gamma(X,\cdot)} Ab \text{ by } \mathcal{F} \longrightarrow \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$$

$\Gamma(X, \cdot)$ is a left exact functor.

Definition 7.18. An **injective object** in an abelian category C is an object I with the following property: Given $0 \longrightarrow A \longrightarrow B$ and $A \longrightarrow i$, there exists an extension $B \longrightarrow I$ that makes the diagram commute.



$\mathrm{Hom}_C(\cdot, I)$ is right exact and Hom is always a left exact, contravariant functor. So given a short exact sequence:

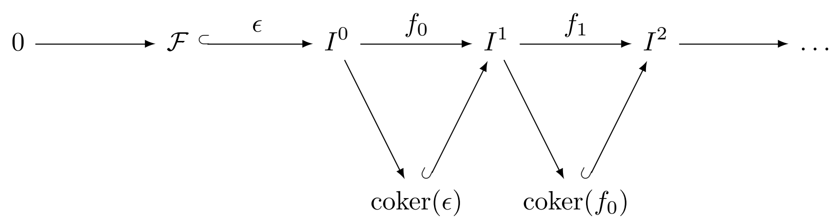
$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \text{ and } C \longrightarrow I$$

We get the exact sequence (where the final arrow follows from the injectivity of I):

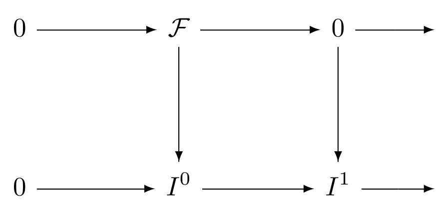
$$0 \longrightarrow \mathrm{Hom}(C, I) \longrightarrow \mathrm{Hom}(B, I) \longrightarrow \mathrm{Hom}(A, I) \longrightarrow 0.$$

How to Compute the right derived functors of a given left exact covariant functor Γ
To compute $R^p\Gamma(\mathcal{F})$ for $\mathcal{F} \in Ob(C)$:

(1) Take an injective resolution of $\mathcal{F}$ (in the category C)



This can be done if every object of C embeds in an injective object of C, i.e. C has "enough injectives."



This diagram is denoted by: $\mathcal{F} \longrightarrow I^\cdot$. We can replace $\mathcal{F}$ by this complex of injectives.

(2) Apply the functor Γ to $I^\cdot$:

$$0 \longrightarrow \Gamma(I^0) \longrightarrow \Gamma(I^1) \longrightarrow \Gamma(I^2) \longrightarrow \dots$$

(3) $R^p\Gamma(\mathcal{F}) = H^p(\Gamma(I^\cdot))$.
Check that $R^0(\Gamma(\mathcal{F})) = \Gamma(\mathcal{F})$:

$$\begin{aligned} R^0(\Gamma(\mathcal{F})) &= H^0(0 \longrightarrow \Gamma(I^0) \longrightarrow \Gamma(I^1) \longrightarrow \dots) \\ &= Ker(\Gamma(I^0) \longrightarrow \Gamma(I^1)) \\ &= \Gamma(\mathcal{F}) \text{ by left exactness of } \Gamma \end{aligned}$$

Given a short exact sequence: $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow$ and injective resolutions $I^{\cdot}, J^{\cdot}$ of A, C respectively, we get an injective resolution $I^{\cdot} \oplus J^{\cdot} = (I \oplus J)^{\cdot}$ of B.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^1 & \longrightarrow & I^1 \oplus J^1 & \longrightarrow & J^1 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^0 & \longrightarrow & I^0 \oplus J^0 & \longrightarrow & J^0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Then we apply the functor Γ to $(I \oplus J)^{\cdot}$ to get an exact diagram.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Gamma(I^1) & \longrightarrow & \Gamma(I^1 \oplus J^1) & \longrightarrow & \Gamma(J^1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Gamma(I^0) & \longrightarrow & \Gamma(I^0 \oplus J^0) & \longrightarrow & \Gamma(J^0) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Gamma(A) & \longrightarrow & \Gamma(B) & \longrightarrow & \Gamma(C) & \longrightarrow & 0
\end{array}$$

We can find the cohomology of this complex and, by using the snake lemma, we get maps $H^p(J^{\cdot}) \longrightarrow H^{p+1}(I^{\cdot})$, which gives us a long exact sequence of cohomology:

$$0 \longrightarrow H^0(\Gamma(I^{\cdot})) \longrightarrow H^0(\Gamma((I+J)^{\cdot})) \longrightarrow H^0(\Gamma(J^{\cdot})) \longrightarrow H^1(\Gamma(I^{\cdot})) \longrightarrow \dots$$

8. Sheaf Cohomology

Definition 8.1. Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X. The **sheaf cohomology** of $\mathcal{F}$ is the right derived functor of $\Gamma(X, \cdot)$. Explicitly: Given $\mathcal{F}$, take a resolution by injective sheaves:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \dots$$

Each $\mathcal{I}^p$ is a sheaf of abelian groups and

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & B \\ & & \downarrow & \swarrow\!\!\!\!\dashrightarrow & \\ & & \mathcal{I}^p & & \end{array}$$

$$H^p(X,\mathcal{F}) = H^p(\Gamma(\mathcal{I}^0 \longrightarrow \Gamma(\mathcal{I}^1 \longrightarrow \Gamma(\mathcal{I}^2 \longrightarrow \dots) = p^{\text{th}}\text{sheaf cohomology of } \mathcal{F}$$

For this to make sense, we need every sheaf of abelian groups to embed into an injective sheaf of abelian groups.

Example 8.2. Let X be a scheme and $\mathcal{F}$ be a quasi-coherent sheaf. By definition: think of $\mathcal{F}$ as a sheaf of abelian (forget the extra $\mathcal{O}_X$-module structure). Compute $H^p(X,\mathcal{F}) = H^p(\Gamma(\mathcal{I}^{\cdot}))$, where $\mathcal{I}^{\cdot}$ is an injective resolution by injective sheaves of abelian groups. It turns out that the category of quasi-coherent sheaves on X also has enough injectives. If you resolve $\mathcal{F}$ by injective quasi-coherent $\mathcal{O}_X$-modules, you get the same result, i.e. the same $H^p(X,\mathcal{F})$.

Cohomology Blackbox

Given any quasi-coherent sheaf $\mathcal{F}$ on a scheme X (or more generally a sheaf of abelian groups on a topological space), we have cohomology groups $H^p(X,\mathcal{F}), \forall p \geq 0$ with the following properties:

(1) $H^0(X,\mathcal{F}) = \Gamma(\mathcal{F})$
(2) Given a short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ we get a long exact sequence

$$H^{p-1}(X,C) \xrightarrow{\delta^{p-1}} H^p(X,A) \longrightarrow H^P(X,C) \xrightarrow{\delta^p} H^{p+1}(X,A),$$

where the connecting morphisms δ can be derived using the snake lemma.
(3) If X is an affine scheme and $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, then $H^p(X,\mathcal{F}) = 0, \forall p > 0$. The converse is also true if we restrict to noetherian schemes (proof from Serre).
(4) Cohomology can be computed using the "Cech complex," i.e. cover X by open sets with trivial cohomology along with something else that we will learn later. In particular, $H^p(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$ can be explicitly be computed in this way.
(5) If $X = \text{Proj } S$, where $S = \frac{A[x_0,\dots,x_n]}{I}$ ($\deg x_i = 1$), and $\mathcal{F}$ is a coherent sheaf, then

$$H^p(X,\mathcal{F}) \text{ is finitely generated as } A\text{-modules}\forall p$$

Lecture 17. March 12, 2009

Remark 8.3. A short exact sequence of sheaves $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact on stalks, but not necessarily exact for all open sets.

Recall that $H^p(X,\mathcal{F}) = p^{\text{th}}$ right derived functor of the global section functor. It is difficult to find and show that injective resolutions are in fact injective, so this type of cohomology is difficult to compute.

Practical Ways to Compute Cohomology: $H^p(X,\mathcal{F})$

(1) Use long exact sequences to relate $H^p(X,\mathcal{F})$ to cohomologies we know
(2) Čech cohomology (Serre's original definition of cohomology)
(3) Instead of using injective resolutions, use more practical resolutions.

Definition 8.4. A sheaf $\mathcal{J}$ of abelian groups on a topological space X is **acyclic for the global section functor** (or **Γ-acyclic**) if $H^p(X,\mathcal{J}) = 0, \forall p > 0$.

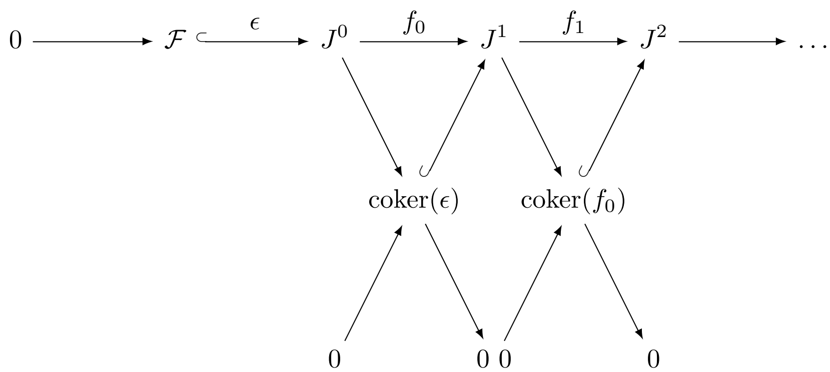
Example 8.5. An injective sheaf is always Γ-acyclic, since it is its own injective resolution: $0 \longrightarrow I \longrightarrow I^0 \longrightarrow 0 \longrightarrow \ldots$ so that $H^p(0 \longrightarrow \Gamma(I^0) \longrightarrow \Gamma(0) \longrightarrow \ldots) = 0, \forall p > 0$

Proposition 8.6. *$H^p(X, \mathcal{F})$ can be computed from any Γ-acyclic resolution of $\mathcal{F}$. More precisely,*

$$H^p(X, \mathcal{F}) = H^p(0 \longrightarrow \Gamma(J^0) \longrightarrow \Gamma(J^1) \longrightarrow \ldots)$$

where $0 \longrightarrow \mathcal{F} \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \ldots$ is a resolution of $\mathcal{F}$ by Γ-acyclic sheaves.

Proof. Given a resolution of Γ-acyclics:



Use induction on p:
$p = 0$: $H^0(X, \mathcal{F}) = \ker(\Gamma(J^0) \longrightarrow \Gamma(J^1))$ by left exactness
$p = 1$: From the short exact seqence:

$$0 \longrightarrow \mathcal{F} \longrightarrow J^0 \longrightarrow K^0 \longrightarrow 0,$$

we get the long exact sequence:

$$0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, J^0) \longrightarrow H^0(X, K^0) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, H^0) = 0$$

$$\begin{aligned} H^1(X, \mathcal{F}) &= \text{coker}(H^0(X, J^0) \longrightarrow H^0(X, K^0) \\ &= \frac{H^0(X, K^0)}{\text{Im}(H^0(J^0) \longrightarrow H^0(K^0))} \\ &= \frac{\ker(\Gamma(J^1) \longrightarrow \Gamma(J^2)}{\text{Im}(\Gamma(J^0) \longrightarrow \Gamma(J^1))} \\ &= H^1(0 \longrightarrow \Gamma(J^0) \longrightarrow \Gamma(J^1) \longrightarrow \Gamma(J^2) \longrightarrow \ldots) \end{aligned}$$

$p \geq 2$: Use the short exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow J^0 \longrightarrow K^0 \longrightarrow 0$, $H^q(X, J^0) = 0$ for $q > 0$ (since J is acyclic), and the snake lemma to get:

$$\ldots \longrightarrow H^{p-1}(X, J^0) = 0 \longrightarrow H^{p-1}(X, K^0) \longrightarrow H^p(X, \mathcal{F}) \longrightarrow H^p(X, J^0) = 0 \longrightarrow \ldots$$

By induction, $H^{p-1}(X, K^0)$ can be computed from a Γ-acyclic resolution of K^0. Any Γ-acyclic resolution of K^0 extends to a Γ-acyclic resolution of $\mathcal{F}$ so our induction is complete. □

More generally, if $C \longrightarrow \Gamma C$ is any (additive) left covariant functor between abelian categories you can compute $R^p(\Gamma(\mathcal{F}))$ from any resolution of Γ-acyclic objects:

$$0 \longrightarrow \mathcal{F} \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \ldots.$$

Definition 8.7. A sheaf $\mathcal{F}$ on a topological space X is **flasque** (or flabby) if $\forall U \subseteq U'$ both open in X, $\mathcal{F}(U') \xrightarrow{\text{restrict}} \mathcal{F}(U)$ is surjective.

Examples of Sheaves of Abelian Groups acyclic for global section functor:

(1) Flasque sheaves are Γ-acyclic
(2) Let X be a smooth manifold, C^∞_X is acyclic (but not flasque).
(3) $\mathcal{A}^p_X =$ (Smooth p-forms on X) are Γ-acyclic, where X is a smooth manifold
(4) Philosophically, any sheaf that admits a partition of unity.

Theorem 8.8. *Let M be a smooth manifold, $H^p_{DR}(M) \cong H^p(M, \underline{\mathbb{R}})$ (where $\underline{\mathbb{R}}$ is the sheaf of locally constant $\mathbb{R}$-valued functions on M).*

Proof.

$$(0 \longrightarrow C^\infty_M \xrightarrow{d} \mathcal{A}^1_M \xrightarrow{d} \mathcal{A}^2_M \xrightarrow{d} \dots)$$

This sequence is exact at stalks (small enough open sets - on contractible sets). By the Poincare lemma, if U is contractible, this is exact (every closed p-form on U is exact). So we can compute sheaf cohomology from the DeRham resolution. □

9. Čech Cohomology

Let X be a topological space and $\mathcal{F}$ a quasi-coherent sheaf of abelian groups. For a fixed open cover U, the Čech cohomology groups are $\check{H}^p(U, \mathcal{F})$ (this depends on the cover).
Point: If X is a noetherian separated scheme (or projective over A), then $\check{H}^p(U, \mathcal{F})$ are all the same for any affine open cover of X and all are also isomorphic to $H^p(X, \mathcal{F})$.

Example 9.1. Let $X = U_0 \cup U_1$ and let $U = \{U_0, U_1\}$ be the open cover of X. Let $\mathcal{F}$ be a sheaf of abelian groups. We want to look at the Čech complex, $\check{C}^\cdot(U, \mathcal{F})$.

$$0 \longrightarrow \check{C}^0(U, \mathcal{F}) = \mathcal{F}(U_0) \times \mathcal{F}(U_1) \xrightarrow{\check{d}^0} \check{C}^1(U, \mathcal{F}) = \mathcal{F}(U_0 \cap U_1) \longrightarrow 0$$

Where $\check{d}^0 : (s_0, s_1) \mapsto s_0|_{U_0 \cap U_1} - s_1|_{U_0 \cap U_1}$ and $\ker(\check{d}^0) = \mathcal{F}(X)$. $\check{H}^1(U, \mathcal{F}) =$ cohomology of $\check{C}(U, \mathcal{F})$ at the first spot = cokernel of $\check{d}^0$.

Example 9.2. Let $X = S^1$ and $\mathcal{F} = \underline{\mathbb{Z}}$. Let U_0 be the upper hemisphere of S^1 and U_1 be the lower hemisphere (where the two sets overlap on an open set). Let U denote this covering of X. As in the previous example, we get the sequence:

$$0 \longrightarrow \underline{\mathbb{Z}}(U_0) \times \underline{\mathbb{Z}}(U_1) \longrightarrow \underline{\mathbb{Z}}(U_0 \cap U_1) \longrightarrow 0,$$

which simplifies to:

$$0 \longrightarrow \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z} \text{ where } (n, m) \mapsto (n - m, n - m)$$

So $\check{H}^0(U, \underline{\mathbb{Z}}) \cong \mathbb{Z}$ and $\check{H}^1(U, \underline{\mathbb{Z}}) \cong \mathbb{Z}$.

Let X be a topological space, $\mathcal{F}$ a quasi-coherent sheaf on X, and $U = \{U_i\}_{i \in I}$ an arbitrary open cover of X, where I is a well-ordered set. More generally, we want to define the Čech cohomology of the sheaf $\mathcal{F}$:

$$0 \longrightarrow \check{C}^0(U, \mathcal{F}) \longrightarrow \dots \longrightarrow \check{C}^{p-1}(U, \mathcal{F}) \xrightarrow{\check{d}^{p-1}} \check{C}^p(U, \mathcal{F}) \longrightarrow \dots$$

$$\text{Where} \qquad \check{C}^p(U, \mathcal{F}) = \prod_{i_0 < \dots < i_p} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_p}) \qquad \text{.for all } p \geq 0.$$

$$\prod_{i \in I} \mathcal{F}(U_i) \longrightarrow \prod_{i_0 < i_1} \mathcal{F}(U_{i_0} \cap U_{i_1}) \qquad \text{is given by:}$$

$$(0, \dots, 0, s_i, 0, \dots, 0) \mapsto (0, \dots, 0, (-1)^\sigma s_i|_{U_i \cap U_j}, 0, \dots, 0),$$

where σ is the permutation that reorders i and j. For $\alpha \in \check{C}^p(U, \mathcal{F})$,

$$\alpha = (\alpha_{i_0 \dots i_p}) \xrightarrow{\check{d}^p} \left(\check{d}^p(\alpha)\right)_{i_0 \dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0 \dots \hat{j} \dots i_{p+1}}|_{U_{i_0} \cap \dots \cap U_{i_{p+1}}}$$

Example 9.3. Let $X = \mathbb{P}^2$ with sheaf $\mathcal{O}_{\mathbb{P}^2}$ and U_0, U_1, U_2 the standard affine cover of X.

$$0 \longrightarrow \check{C}^0(U, \mathcal{O}_{\mathbb{P}^2}) \xrightarrow{f} \check{C}^1(U, \mathcal{O}_{\mathbb{P}^2}) \xrightarrow{g} \check{C}^2(U, \mathcal{O}_{\mathbb{P}^2}) \longrightarrow \check{C}^3(U, \mathcal{O}_{\mathbb{P}^2}) = 0$$

Where:

$$\check{C}^0(U, \mathcal{O}_{\mathbb{P}^2}) = \mathcal{O}_{\mathbb{P}^2}(U_0) \times \mathcal{O}_{\mathbb{P}^2}(U_1) \times \mathcal{O}_{\mathbb{P}^2}(U_2)$$
$$\check{C}^1(U, \mathcal{O}_{\mathbb{P}^2}) = \mathcal{O}_{\mathbb{P}^2}(U_1 \cap U_2) \times \mathcal{O}_{\mathbb{P}^2}(U_0 \cap U_2) \times \mathcal{O}_{\mathbb{P}^2}(U_0 \cap U_1)$$
$$\check{C}^2(U, \mathcal{O}_{\mathbb{P}^2}) = \mathcal{O}_{\mathbb{P}^2}(U_0 \cap U_1 \cap U_2)$$

The maps are given by:

$$f : (s_0, s_1, s_2) \mapsto (s_1 - s_2, s_0 - s_2, s_0 - s_1) \text{ and } g : (t_{12}, t_{02}, t_{01}) \mapsto t_{12} - t_{02} + t_{01}.$$

Using Čech cohomology with respect to the standard cover of $\mathbb{P}^n_k$, we can explicitly compute $H^p(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(m)), \forall p, n, m$. Also, using Hartshorne, we can prove that Čech cohomology of any open cover agrees with sheaf cohomology.

Lecture 18. March 17, 2009

Recall: sheaf cohomology can be computed from other resolutions (i.e. flasque).

Lemma 9.4 (Important)**.** *If $Y \overset{i}{\hookrightarrow} X$ is a closed embedding, and $\mathcal{F}$ is a sheaf of abelian groups on Y, then*

$$H^p(X, i_*\mathcal{F}) = H^p(Y, \mathcal{F}).$$

Proof. To compute $H^p(Y, \mathcal{F})$, take a flasque resolution of $\mathcal{F}$:

$$0 \longrightarrow \mathcal{F} \longrightarrow J^0 \longrightarrow J^1 \longrightarrow J^2 \longrightarrow \dots$$

By definition, $H^p(Y, \mathcal{F}) = H^p(0 \longrightarrow \Gamma(Y, J^0) \longrightarrow \Gamma(Y, J^1) \longrightarrow \dots)$. Apply i_* to the resolution above. Then:

$$\dagger \quad 0 \longrightarrow i_*\mathcal{F} \longrightarrow i_*J^0 \longrightarrow i_*J^1 \longrightarrow i_*J^2 \longrightarrow \dots$$

is still an exact as a complex of sheaves on X because it is exact at stalks: if $p \in Y$ then looking at the stalks of the complex at p recovers the original resolution with stalks at p and if $p \notin Y$ then the stalks of all $(i_*J^k)_p = 0$. In fact, this is still a flasque resolution: $V \subseteq U \subseteq X$: $i_*J^k(U) = J^k(U \cap Y) \xrightarrow{\text{restriction}} i_*J^k(V) = J^k(V \cap Y)$ is surjective since the original resolution on Y was flasque. So we can compute $H^p(X, i_*\mathcal{F})$ from †. Take global section:

$$H^p\big(0 \longrightarrow \Gamma(X, i_*J^0) \longrightarrow \Gamma(X, i_*J^1) \longrightarrow \Gamma(X, i_*J^2) \longrightarrow \dots\big)$$

But $\Gamma(X, i_*J^p) = i_*J^p(X) = J^p(X \cap Y) = J^p(Y) = \Gamma(Y, J^p)$. □

Theorem 9.5. *Let $S = A[x_0, \dots, x_n]$ where A is a noetherian ring and the degree of each x_i is 1. Let $X = Proj\ S$. Formally, $S = A[x_0, \dots, x_n] = Sym^{\cdot}_A(M)$, where M is the free A-module on $x_0, \dots, x_n$. Then:*

(1) *The natural map $S \longrightarrow \Gamma_*(X, \mathcal{O}_X) = \oplus_{m \in \mathbb{Z}} H^0(X, \mathcal{O}_X(m))$, which is given by s (of degree m) $\mapsto s \in H^0(X, \mathcal{O}_X(m))$ where $\frac{s}{1} \in \mathcal{O}_X(D_+(x_i)) = (S\big[\frac{1}{x_i}\big])_m$, is an isomorphism. In part:*

$$H^0(X, \mathcal{O}_X(m)) = \begin{cases} 0 & \text{if } m < 0 \\ S_m & \text{if } m \geq 0 \end{cases}$$

(2) *$H^p(X, \mathcal{O}_X(m)) = 0, \forall m$ and $\forall p$ satisfying $0 < p < n$ or $p > n$*
(3) *$H^n(X, \mathcal{O}_X(-m)) \cong (H^0(X, \mathcal{O}_X(-m-n-1)))^* = (Sym^{(m-n-1)}M)^*$ and when $m = n+1$ this simplifies to: $H^n(X, \mathcal{O}_X(-n-1)) \cong A$*

(4) *The natural map*

$$H^0(X, \mathcal{O}_X(m)) \times H^n(X, \mathcal{O}_X(-m-n-1)) \longrightarrow H^n(X, \mathcal{O}_X(-n-1)) = A$$

is a perfect pairing of free A-modules.

Recall that $\mathcal{O}_X(m) = \widetilde{S(m)}$ and
$S_m =$ free A-modules of rank $\binom{n+m}{m} = Sym_A^m(A \otimes \cdots \otimes A) = Sym_A^m((M))$.

Application: Say $C \subseteq^i \mathbb{P}_k^2$ is an irreducible curve of degree d given by an irreducible homogeneous element $F \in k[x, y, z]$ of degree d. Compute $H^1(C, \mathcal{O}_C)$ (this is called the genus of C). By the previous lemma, $H^1(C, \mathcal{O}_C) = H^1(\mathbb{P}_k^2, i_*\mathcal{O}_C)$. We have a short exact sequence: (where I_C is the ideal sheaf)

$$0 \longrightarrow I_C \longrightarrow \mathcal{O}_{\mathbb{P}_k^2} \longrightarrow i_*\mathcal{O}_C \longrightarrow 0$$

$I_C = (\tilde{F}) = \widetilde{S(-d)}, \mathcal{O}_{\mathbb{P}_k^2} = (\tilde{S}),$ and $i_*\mathcal{O}_C = \widetilde{S/(F)}$ so we get the short exact sequence:

$$0 \longrightarrow \widetilde{S(-d)} \xrightarrow{F} \tilde{S} \longrightarrow \widetilde{S/(F)} \longrightarrow 0$$

And we get a long exact sequence of cohomology on $\mathbb{P}_k^2$:

$$0 \longrightarrow H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d)) \longrightarrow H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \longrightarrow H^0(C, \mathcal{O}_C) \longrightarrow H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d))$$

$$\longrightarrow H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \longrightarrow H^1(C, \mathcal{O}_C) \longrightarrow H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d)) \longrightarrow H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \longrightarrow \dots$$

$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d)) = 0, H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = k, H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d)) = 0$ where the first and third come from the previous theorem. So $0 \longrightarrow k \longrightarrow H^0(C, \mathcal{O}_C) \longrightarrow 0$ which implies that $H^0(C, \mathcal{O}_C) = k$.

$$0 \longrightarrow H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0 \longrightarrow H^1(C, \mathcal{O}_C) \xrightarrow{\cong} H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d)) \longrightarrow H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$$

$H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-m-3))$ is dual to $H^0(p^2, \mathcal{O}_{\mathbb{P}^2}(m))$ and $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(0))$ is dual to $H^0(p^2, \mathcal{O}_{\mathbb{P}^2}(-3))$.
So that $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d))$ is dual to $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-3)) = \big[k[x, y, z]\big]_{d-3} = \begin{cases} \binom{d-1}{2} & \text{if } d \geq 3 \\ 0 & \text{if } d < 3 \end{cases}$.

Therefore the genus of C, $g(c) = \begin{cases} \binom{d-1}{2} = \frac{(d-1)(d-2)}{2} & \text{if } d \geq 3 \\ 0 & \text{if } d < 3 \end{cases}$.

Proof. (Of Theorem of Cohomology $H^p(\mathbb{P}_A^n = X, \mathcal{O}_X(m))$)

(1) Cover $\mathbb{P}_A^n$ by the standard affine open cover $D_+(x_i) = \text{Spec }\big((S[\frac{1}{x_i}])_0\big)$. Look at the Čech complex for $\mathcal{O}_X(m) = \widetilde{S}(m)$.

$$\begin{aligned} \mathcal{O}_X(m)(D_+(x_{i_0}) \cap \cdots \cap D_+(x_{i_t})) &= \mathcal{O}_X(m)(D_+(x_{i_0} \dots\ x_{i_t}) \\ &= [S(m)[\frac{1}{x_{i_0} \dots x_{i_t}}]]_0 \\ &= S[\frac{1}{x_{i_0} \dots x_{i_t}}]_m \end{aligned}$$

It will be the m^{th} graded piece of the chain complex.

$$0 \longrightarrow S[\frac{1}{x_0}] \times \cdots \times S[\frac{1}{x_n}] \xrightarrow{\check{d}^0} \oplus_{i<j} S[\frac{1}{x_i x_j}] \longrightarrow \cdots$$

$$\longrightarrow \oplus_{i<j<k} S[\frac{1}{x_i x_j x_k}] \longrightarrow \oplus_i S[\frac{1}{x_0 \dots \hat{x_i} \dots x_n}] \xrightarrow{\check{d^{n-1}}} S[\frac{1}{x_0 \dots x_n}] \longrightarrow 0$$

$$\text{where } \alpha = (\frac{s_0}{x_0^r}, \dots, \frac{s_n}{x_n^r}) \mapsto (\frac{s_0}{x_0^r} - \frac{s_1}{x_1^r}, \frac{s_0}{x_0^r} - \frac{s_2}{x_2^r}, \dots)$$

(note that we have to pay attention to the signs in the above map). $\alpha \in \ker(\check{d}^0) \Rightarrow \frac{s_0}{x_0^r} = \frac{s_1}{x_1^r}$ in $S[\frac{1}{x_0 x_1}] \Rightarrow x_1^r s_0 = x_0^r s_1 \Rightarrow s_0 = x_0^r s, s_1 = x_1^r s$ for some $s \in S \Rightarrow \frac{s_0}{x_0^r} = \frac{s}{1} \frac{s_1}{x_1^r} = \frac{s}{1}$.

(2) See Hartshorne.

(3) Compute $H^n(X, \mathcal{O}_X(m)) =$ the degree m part of the coker$(\check{d^{n-1}})$. A free A-module basis of $S[\frac{1}{x_0 \dots x_n}]$ is $x_0^{a_0} \dots x_n^{a_n}$, where $a_i \in \mathbb{Z}$.

$$\oplus_{i=0}^n \mathcal{O}_X(D_+(x_0) \cap \cdots \cap D_(x_n)) \longrightarrow \mathcal{O}_X(m)(D_+(x_0) \cap \cdots \cap D_+(x_n)) \longrightarrow 0$$

$$\oplus_{i=0}^n S[\frac{1}{x_0 \dots \hat{x_i} \dots x_n}] \longrightarrow S[\frac{1}{x_0 \dots x_n}] \longrightarrow 0 \text{ degree } m \text{ part for } \mathcal{O}_X(m)$$

$$(0, \dots, 0, \frac{s_i}{(x_0 \dots \hat{x_i} \dots x_n)^t}, 0, \dots, 0) \mapsto (-1)^i \frac{s}{(x_{i0} \dots x_{in})^t}$$

The image of $\check{d}^{n-1}$ is a free A-module spanned by $x_0^{a_0} \dots x_n^{a_n}$, where not all of the a_i are less than the degree $-n-1$ in $S[\frac{1}{x_0 \dots x_n}]$ but not in the image of $\check{d^{n-1}}$. So $H^n(X, \mathcal{O}_X(m))$ is a free A-module on $\{x_0^{a_0} \dots x_n^{a_n} \mid \sum a_i = m, a_i < 0\}$. So $H^n(X, \mathcal{O}_X(-n-1)) = A x_x^{-1} \dots x_n^{-1}$.

Perfect Pairing:

$$H^0(X, \mathcal{O}_X(m-n-1)) \times H^n(X, \mathcal{O}_X(-m)) \longrightarrow H^n(X, \mathcal{O}_X(-n-1)) = A$$

$$\text{Basis over } A : \{x_0^{b_0} \dots x_n^{b_n}\} \times \{x_0^{a_0} \dots x_n^{a_n}\} \mapsto \{x_0^{a_0+b_0} \dots x_n^{a_n+b_n}\}$$

$$\text{where } \sum b_i = m-n-1, b_i \geq 0 \text{ and } \sum a_i = -m, a_i < 0 \qquad \sum (a_i + b_i) = -n-1$$

This generated cokernel of $\check{d^{n-1}}$ in the degree $-n-1$.

(4) See Hartshorne for the proof. The perfect pairing is just multiplication of classes in $H^n(X, 0_X(m))$ represented by an element $\sum_{\sum a_i = -m-n-1} b_i x_0^{a_0} \dots x_n^{a_n}$ and by element $\sum_{|I|=m} c_j X_I$, where $b_i, c_j \in A$.

□

Lecture 19. March 19, 2009

Duality in C is a special case of Serre Duality (which is a special case of Grothendieck duality).

Context for most Algebraic Geometers:

- X is a smooth, projective variety over $k = \overline{k}$ (or $X = \text{Proj } (\frac{k[x_0, \dots, x_n]}{I}$.
- $\Omega_{X/k}$ is locally free of rank d (i.e. if at p we have $z_1, \dots, z_d$ are parameters, then Ω_X is a free $\mathcal{O}_X$-module generated by $dz_1, \dots, dz_n$ in a neighborhood of p).
- $\omega_X = \wedge^d \Omega_X$ is locally free of rank 1 (local generator $dz_1 \cap \cdots \cap dz_d$)

Serre Duality: Let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module of finite rank. $\mathcal{F}^v = Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. There's a perfect pairing:

$$H^p(X, \mathcal{F}) \times H^{d-p}(X, \mathcal{F}^v \otimes \omega_X) \longrightarrow H^d(X, \omega_X) \xrightarrow{\cong} k,$$

i.e. $H^p(X, \mathcal{F})$ is the dual over k to $H^{d-p}(X, \mathcal{F}^v \otimes \omega_X)$.

$\mathcal{O}_X(-m)$ has rank 1 it is a locally free $\mathcal{O}_X$-module $\mathcal{L}$, $\mathcal{L}^v = \mathcal{O}_X(m)$
$\mathcal{O}_X(m-n-1) = \mathcal{O}_X(m) \otimes \mathcal{O}_X(-n-1) = \mathcal{L}^v \otimes \omega_{\mathbb{P}^n}$, $\mathcal{O}_X(-n-1)$ is the canonical module on $\mathbb{P}^n$. The case where $X = \mathbb{P}^n_k, \mathcal{F} = \mathcal{O}_X(-m), p = n$ is the duality in part (c) of the previous theorem.

Part 4. Divisors and All That

10. Basics of Divisors

Hartshorne II, §6
Assumption †: X **is a noetherian separated integral scheme, regular in codimension 1.** Each condition has the following implications:
noetherian: X can be covered by finitely many affine open sets each of which is the Spec of a noetherian ring
integral: $\forall U \subseteq X$ open affine where $U = \text{Spec } A$, $\mathcal{O}_X(U)$ is a domain $\Rightarrow X$ has a function field K:

$$\begin{aligned} K &= frac(\mathcal{O}_X(U)), \forall U \subseteq X \text{ open} \\ &= frac(A) \\ &= \mathcal{O}_{X,\eta}, \text{ where } \eta \text{ is the generic point of } X \end{aligned}$$

regular in codimension 1: Let $y \in X$ be any generic point of a codimension 1 closed irreducible subscheme. If $y \in U = \text{Spec } A \subseteq X$ and $p \subseteq \text{Spec } A$ of height 1, then $\mathcal{O}_{X,y} = A_p$ is a DVR.

Main Case: X is an irreducible, smooth (or normal), projective variety over $k = \overline{k}$
$X = \text{Proj } S$, where $S = \frac{k[x_0, \ldots, x_n]}{I}$, normal domain.

Definition 10.1. A **prime divisor** on X is a codimension 1 closed integral subscheme of X.

Definition 10.2. Let K be the function field of X, $Y \subseteq X$ a closed subscheme of codimension 1, and $y \in Y$ a generic point. So that $K \supseteq \mathcal{O}_{X,Y} = \mathcal{O}_{X,y}$ is a DVR. For $f \in K^* = K \setminus \{0\}$, the **"order of vanishing of f along Y"**, $ord_Y(f)$, is

$ord_Y(f) =$ order of f in $\mathcal{O}_{X,y} = n$, where $(t) \subset \mathcal{O}_{X,y}$ maximal ideal and $f = (\text{unit in } \mathcal{O}_{X,Y}) \cdot t^n$.

Since K is a discrete valuation ring, every element of K can be written as $(unit) \cdot t^n$ for some unique $n \in \mathbb{Z}$ where (t) is the maximal ideal of the discrete valuation ring $\mathcal{O}_{X,Y}$.

Definition 10.3. A **(Weil) divisor** on X is an element of the free abelian group $DivX$ on the prime diviors of X.

Proposition 10.4. *There is a homomorphism of abelian groups:*

$$\begin{gathered} K^* = K \setminus \{0\} \xrightarrow{div} DivX \\ f \mapsto div(f) = \sum_{Y \text{ prime divisor}} ord_Y(f) \cdot Y \\ f \cdot g \mapsto div(f) + div(g) \end{gathered}$$

This is the valuation ring of X in K.

Proof. This is effectively the same proof as in 631. The proof is in Hartshorne. $\square$

Remark 10.5. Y uniquely determines the valuation "ord_Y". But the separatedness of X says that the "ord_Y" valuation uniquely determines Y. Let y be the generic point of Y.

$$\begin{array}{ccc} \text{Spec } K & \xrightarrow{(0) \mapsto y} & X \\ \big\downarrow & & \big\downarrow \\ \text{Spec } (\mathcal{O}_{X,Y}) & \longrightarrow & \text{Spec } \mathbb{Z} \end{array}$$

Separatedness of X gives us the uniqueness of the map Spec $(\mathcal{O}_{X,Y}) \longrightarrow X$.

Definition 10.6. The group of **principal divisors**, $P(X)$, is the image of div in $DivX$.

Definition 10.7. The **class group of** X is the cokernel of div, i.e. $Cl(X) = \frac{DivX}{P(X)}$.

Definition 10.8. The **degree of a divisor**, $D = \sum n_i Y_i$, is $\deg(D) = \deg(\sum n_i Y_i) = \sum n_i \deg Y_i = \sum n_i d_i$, where $Y_i = \mathbb{V}(F_i)$ and $d_i = \deg(Y_i) = \deg(F_i)$.

Example 10.9. $Cl(\mathbb{P}^n_k), k$ a field and $\mathbb{P}^n_k = \text{Proj } k[x_0, \ldots, x_n]$.

Y prime divisor $\Leftrightarrow Y = \mathbb{V}(F)$ and F is a homogeneous, irreducible polynomial

$Div(\mathbb{P}^n_k) \xrightarrow{\deg} \mathbb{Z}$ has kernel

$P(\mathbb{P}^n_k) = \{D = \sum n_i Y_i \mid \deg(D) = \sum n_i \deg Y_i = \sum n_i \deg(F_i) = \deg(div(F_1^{n_1} \ldots F_t^{n_t}))\}$,

$F_1^{n_1} \ldots F_t^{n_t} \in k(\mathbb{P}^n_k) \quad \Rightarrow \quad Cl(\mathbb{P}^n_k) \cong \mathbb{Z}$.

Proposition 10.10. *Let X be a scheme satisfying $\dagger$, $Z \subseteq X$ a closed subscheme, and $U = X - Z$. There's a natural surjective map:*

$$Cl(X) \longrightarrow\!\!\!\!\rightarrow Cl(X - Z) = Cl(U) \text{ given by } \sum n_i Y_i \mapsto \sum n_i (Y_i \cap U)$$

If Z has codimension atleast 2, this is an isomorphism. If Z is integral of codimension 1, then the kernel is generated by $\mathbb{Z}$.

Lecture 20. March 24, 2009

Exercise 10.11. In chapter II: think through 5.17 (but don't write-up) in order to do 5.18, 6.2ab,6.6, 6.8. In chapter II: 4.5.

Continue to assume that X is noetherian, integral, separated, and regular in codimension 1. The integral assumption gives us a function field, K. Regular in codimension 1 gives us that $\forall Y \subseteq X$ of codimension 1, $\mathcal{O}_{X,Y}$ is a DVR. Since K is a discrete valuation ring, every element of K can be written as (unit)$\cdot t^n$ for some unique $n \in \mathbb{Z}$ where (t) is the maximal ideal of the DVR $\mathcal{O}_{X,Y}$.

Definition 10.12 (1)**.** A Weil divisor is **Cartier** if it is locally principal. Precisely, X has an open cover $\{U_i\}$ such that $\exists f_i \in K$ such that $D \cap U_i = div_{U_i} f_i$

Think of D as given by $\{U_i, div f_i\}$ where

$$div_{U_i \cap U_j} f_i = (div_{U_i} f_i) \cap U_j = (div_{U_j} f_j) \cap U_i = div_{U_i \cap U_j} f_j$$

So that $div_{U_i \cap U_j}(f_i \cdot f_j^{-1})$ has no zeros or poles $\Leftrightarrow f_i f_j^{-1} \in \mathcal{O}^*_X(U_i \cap U_j)$

Example 10.13. Let $X = \mathbb{P}^n, D =$ codimension 1 integral subscheme (i.e. subscheme given by (x_0)). D is not principal but it is Cartier:

$$D \cap U_i = D \cap D_+(x_i) = div_{U_i}\left(\frac{x_0}{x_i}\right) \text{ and } D \cap U_0 = 0$$

FACT: If X happens to have the property that all local rings $\mathcal{O}_{X,x}$ are UFDs ($\forall x \in X$ closed points), then all Weil divisors are locally principal (this comes down to the fact that height 1 prime ideals in a UFD are principal).
MAIN CASE: X is a smooth variety. In this case, $\mathcal{O}_{X,x}$ is a UFD $\forall x \in X$.

Example 10.14. Let $X = \text{Spec}\left(\frac{k[x,y,z]}{z^2-xy}\right)$ (this is a normal scheme).
Consider $Y =$ closed subscheme given by the height 1 prime (y, z)
Claim: Y is a Weil divisor but it is not Cartier. To show this, look at the closed point $p = (x, y, z)$. (y, z) is not principal in $\mathcal{O}_{X,p} = \frac{k[x,y,z]}{z^2-xy}(x, y, z)$.

$$\begin{aligned} X - Y &= \text{Spec}\left(\frac{k[x,y,z]}{z^2 - xy}\right) - \mathbb{V}(y,z) \\ &= \text{Spec}\left(\frac{k[x,y,z]}{z^2 - xy}\right) - \mathbb{V}(y,z^2) \\ &= \text{Spec}\left(\frac{k[x,y,z]}{z^2 - xy}\right) - \mathbb{V}(y) \\ &= D(y) \\ &= \text{Spec}\left(\frac{k[x,y,z]}{\frac{z^2}{y} - x}\left[\frac{1}{y}\right]\right) \\ &= \text{Spec}\left(k[y,z]\left[\frac{1}{y}\right]\right) \end{aligned}$$

$$\mathbb{Z} \xrightarrow{\epsilon} Cl(X) \twoheadrightarrow Cl(X - Y) = 0 \text{ where } \epsilon(n) = nY$$

$Cl(X - Y) = 0$ because $X - Y$ is Spec of a UFD. Look at y in $\mathcal{O}_{X,Y} = \frac{k[x,y,z]}{xy-z^2}(y, z)$ to show that $div(y) = 2Y$. This implies that $Cl(X) \cong \mathbb{Z}/2\mathbb{Z}$.

Definition 10.15 (2)**.** A **Cartier** divisor is the data $\{U_i, f_i\}$ where $\cup U_i = X$ is an open cover of X, $f_i \in K^*$, where $f_i f_j^{-1} \in \mathcal{O}_X^*(U_i \cap U_j)$.

Definition 10.16 (3)**.** A **Cartier** divisor is a global section of a sheaf $\mathcal{K}^*/\mathcal{O}_X^*$, where $\mathcal{K}$ is the locally constant sheaf on X given by K (unnecessary to include locally).

Remark 10.17. If X is noetherian, integral, separated, and regular in codimension 1 (i.e. our initial assumptions for this lecture), then these three definitions are all (obviously) equivalent. However, definition 3 (definition 2) makes sense on an arbitrary scheme, where $\mathcal{K}$ is the sheaf associated to the presheaf $\mathcal{K}(U) = \mathcal{O}_X(U)[S^{-1}]$, where S is the multiplicative systems of non-zero divisors in $\mathcal{O}_X(U)$.

11. DIVISORS AND INVERTIBLE SHEAVES

Temporarily drop the assumption that X is noetherian, integral, separated, and regular in codimension 1

Definition 11.1. An **invertible sheaf** on a scheme X is a locally free $\mathcal{O}_X$-module of rank 1.

Example 11.2. If $\mathcal{L}$ is invertible, $\mathcal{L}^{-1} = Hom_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is invertible. Note:

$$\mathcal{L} \otimes \mathcal{L}^{-1} \xrightarrow{\cong} \mathcal{O}_X$$

Defining the presheaf: $\forall U \subset X$ open,

$$\mathcal{L}(U) \otimes Hom_{\mathcal{O}_X}(\mathcal{L}(U), \mathcal{O}_X(U)) \longrightarrow \mathcal{O}_X(U) \text{ given by } s \otimes \varphi \mapsto \varphi(s)$$

Check that this is an isomorphism and take the sheaf associated to this presheaf.

If $\mathcal{L}, \mathcal{M}$ are invertible, then $\mathcal{L} \otimes \mathcal{M}$ is also invertible.

Remark 11.3. The set of invertible sheaves on X (up to isomorphism) forms a group, called the **Picard** group of X.

Example 11.4 (Main). Assume (again) that X is noetherian, integral, separated, and regular in codimension 1. Given a (Weil) divisor D, define the corresponding sheaf $\mathcal{L}(D)$ as a subsheaf of $\mathcal{K}$ as (for $U \subseteq X$ non-empty):

$$\mathcal{L}(D)(U) = \{g \in K^* \mid div_U g + (D \cap U) \geq 0\} \cup \{0\} \subseteq K$$

This is an $\mathcal{O}_X(U)$-module: $f \in \mathcal{O}_X(U)$, $div_U f \geq 0$ and $g \in \mathcal{L}(D)(U) \Rightarrow (fg) \in \mathcal{L}(D)(U)$ because $div_U(fg) + (D \cap U) = div_U(f) + div_U(g) + (D \cap U) \geq 0$.
IF D IS CARTIER, THEN $\mathcal{L}$ IS INVERTIBLE!
D Cartier $\Rightarrow$ cover X by U_i such that $D \cap U_i = div_{U_i}(f_i)$

$$\mathcal{L}(D)(U_i) = \{g \in K^* \mid div_{U_i} g + div_{U_i} f \geq 0\} \qquad \Leftrightarrow$$

$$\{g \in K^* \mid div_{U_i}(g \cdot f_i) \geq 0\} = \{g \in K^* g \cdot f_i \in \mathcal{O}_X(U_i)\}$$
$$= \frac{1}{f_i} \cdot \mathcal{O}_X(U_i)$$

Proposition 11.5. *The (group) map:*

$$CaDivX \longrightarrow \{\textit{invertible sheaves on } X\}$$

$$\begin{array}{ccc} D_1 - D_2 & \longrightarrow & \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1} \\ \downarrow & & \downarrow \textit{mod out by } \cong \\ \dfrac{CaDivX}{P(X)} & \longrightarrow & P(X) \end{array}$$

is an isomorphism if X is a noetherian, integral, separated, and regular in codim. 1.

Of isomorphism.
X is integral, s every invertible sheaf is (isomorphic to) a subsheaf of $\mathcal{K}$.

$$\mathcal{O}_X \hookrightarrow K \Rightarrow \mathcal{L} = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X \hookrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K} \cong \mathcal{K}$$

Identify $\mathcal{L}$ with its image in $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K} \cong \mathcal{K}$ so that $\mathcal{L}$ is a subsheaf of $\mathcal{K}$, which is locally free of rank 1. Cover $\mathcal{L}$ by U_i such that $\mathcal{L}|_{U_i} = \frac{1}{f_i} \cdot \mathcal{O}_X|_{U_i}$. Define D by the data $\{U_i, f_i\}$. Since $(\mathcal{L}|_{U_i})|_{U_j} = (\mathcal{L}|_{U_j})|_{U_i}$, $\frac{1}{f_i} \cdot \mathcal{O}_X|_{U_i \cap U_j} = \frac{1}{f_j} \cdot \mathcal{O}_X|_{U_i \cap U_j} \Rightarrow \frac{1}{f_i} = s_{ij} \frac{1}{f_j}$ for some $s_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$. The s_{ij} are called **transition functions**. Note that $\{s_{ij}\} \in \prod_{i,j} \mathcal{O}_X^*(U_i \cap U_j)$. □

Lecture 21. March 26, 2009

Let X be noetherian, integral, separated, and regular in codimension 1.
Let $\eta \in X$ be the generic point. Then $\mathcal{O}_{X,\eta}$ is the *function field* K of X.

$$\{\text{Weil Div}\} \supseteq \{\text{Cartier Div.=Locally principal}\} \supseteq \{\text{Principal Div}\} = P(X)$$

where Weil Divisors equal Cartier Divisors when X is nonsingular. Let $D \in CaDiv$, then $\mathcal{L}(D)$ is in an invertible sheaf. There is a one-to-one corresponence between locally prinicipal divisors and invertible sheafs. Principal divisors are in one-to-one correspondence between the trivial invertible sheafs (i.e. isomorphic to $\mathcal{O}_X$).

$$\begin{array}{ccccc} \{\text{Weil Div}\} & \xhookrightarrow{\supseteq} & \{\text{Cartier Div.}\} & \xhookrightarrow{\supseteq} & P(X) = \{div(f) \mid f \in K^*\} \\ & & \uparrow\downarrow & & \uparrow\downarrow \\ & & \text{invertible sheaf, subsheaf of } K & & \{f \cdot \mathcal{O}_X \mid f \in K\} \end{array}$$

$$\frac{WDiv}{P(x)} = Cl(X) \supseteq \frac{CaDiv}{P(X)} = Pic(X) = \{\text{group of iso. classes of invertible sheaves on } X\}$$

Example 11.6. Let $X = \mathbb{P}^n_A = \text{Proj } S$ where S is an invertible sheaf of $\mathcal{O}_X$-modules and so $\widetilde{S(m)} = \mathcal{O}_X(m)$. Let f be homogeneous.

$$\mathcal{O}_X(m)(D_+(f)) = \left[S(m)\left[\frac{1}{f}\right]\right]_0 = \left[S\left[\frac{1}{f}\right]\right]_m$$

$$\mathcal{O}_X(m)(D_+(x_i)) = \left[S\left[\frac{1}{x_i}\right]\right]_m = x_i^m \mathcal{O}_X(D_+(x_i))$$

Let A be a field (not neccessary, but easier to think about). $Cl(X) \cong \mathbb{Z}$ and for some hyperplane H, $Cl(X) = \mathbb{Z} \cdot H$. Fix $H = \mathbb{V}(x_0)$.

$$\mathcal{O}_X(m) \cong \mathcal{O}_X(mH)$$

$$\mathcal{O}_X(mH)(D_+(x_i)) = \{f \in K \mid div(f) + mH \geq 0\}$$

$f = \frac{G}{F}, G, F$ are homogeneous of the saame degree and $\frac{G}{F} = \frac{G_1^{a_1} \ldots G_r^{a_r}}{F_1^{b_1} \ldots F_s^{b_s}}$.

$$div(f) = a_1 \mathbb{V}(a_1) + \cdots + a_r \mathbb{V}(a_r) - b_1(F_1) - \cdots - b_s(F_s)$$

Need "$+mH$"≥ 0. This forces $F = x_0^i$, where $i \leq m$ and $\deg(G) = \deg(F)$ so wlog,

$$\frac{G}{F} \in \mathcal{O}_X(mH)(X). \quad \Leftrightarrow \quad \frac{G}{F} = \frac{\text{any element of } [S]_m}{x_0^m}.$$

From this we get a map on sheaves:

$$\mathcal{O}_X(m) \xrightarrow{\cong} \mathcal{O}_X(mH) \text{ given by } f \mapsto \left(\frac{f}{x_0^m}\right)$$

Therefore $\mathcal{O}_X(m) \cong \mathcal{O}_X(D)$, where $D \in Cl(\mathbb{P}^n)$ is of the form mH.

TERMINOLOGY:

(1) Another word for an invertible sheaf is a line bundle:
Let L be a line bundle on X.

$$\begin{array}{ccccc} L & \supseteq & f^{-1}(U) & \xrightarrow{\cong} & U \times k \\ \downarrow f & & \downarrow f & & \\ X & \supseteq & U & & \end{array}$$

where $U \ni x \mapsto \hat{f}(x) = (x, s(x)) \in U \times k$. Let $\mathcal{L}$ be the sheaf of sections of $L \longrightarrow X$: $\mathcal{L}(U) = \{U \xrightarrow{s} L \mid f \circ s = Id\}$ so that $\mathcal{L}(U) \cong \mathcal{O}_X(U)$ by the map $x \mapsto s(x)$.

(2) Given an invertible sheaf $\mathcal{L} \leftrightarrow \mathcal{L}(D)$, "take a line-bundle D", "$D_1 + D_2$, "The Chern class of $\mathcal{L}$ is the corresponding divisor class D"

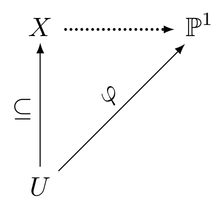
Recall: $X \xrightarrow{f} Y$ is a morphism of schemes. An invertible sheaf $\mathcal{L}$ on Y can be pulled-back to $f^*\mathcal{L}$ ("$f^*\mathcal{L} = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{L}$" by an "abuse" of notation). The pull-back sheaf $f^*\mathcal{L}$ must also be invertible. Think of $\mathcal{L}$ as a Cartier divisor, i.e. $\mathcal{L} = \mathcal{L}(D)$, $\{U_i, f_i\}$, where $f_i \in K^*$. $f^*\mathcal{L}$ should be correspond to the data $\{f^{-1}(U_i), f^*(f_i)\}$. Assuming that $Im(f) \nsubseteq Supp(D)$, we get that $f^*(f_i) \in K(X)$. Without loss of generality, we can replace D by a linearly equivalent D' (which represent the same element of $Pic(X)$) to assume that $Im(f) \nsubseteq Supp(D)$.

Example 11.7 (Classical). $\mathbb{P}^1 \xrightarrow{\nu_3} \mathbb{P}^3$ by $[s : t] \longrightarrow$. Note that s^3, s^2t, st^2, t^3 are not regular functions on $\mathbb{P}^1$. However, they are global sections of the invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(3)$. We could instead have written the map as $[1 : \frac{t}{s} : \frac{t^2}{s^2} : \frac{t^3}{s^3}]$. These are rational functions on $\mathbb{P}^1$ and are all global sections of the sheaf $\mathcal{O}_X(D)$, where $D = 3p$ and $p = [0 : 1]$ since we divided by s^3.

General Picture:
Given any scheme X over A, any invertible sheaf $\mathcal{L}$ on X, and global sections $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$, there is a map of schemes of A:

$$X \dashrightarrow \mathbb{P}^n_A \quad \text{given by } x \mapsto [s_0(x) : \cdots : s_n(x)]$$

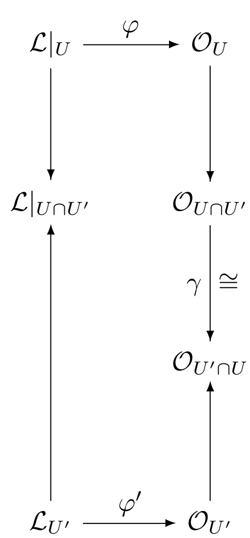
This is a morphism on the open set, U, which is the complement of a common zero set of the s_i's. Every map to $\mathbb{P}^n$ of this type.



$\varphi^*\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{L}|_U$, $\varphi^*(x_i) == s_i$.
How do we make sense out of "$s_i(x)$"?
Choose a neighborhood U of x such that

$$\mathcal{L}|_U \underset{\cong}{\xrightarrow{\varphi}} \mathcal{O}_{\mathbb{P}^n}|_U = \mathcal{O}_U \text{ given by } s_i \mapsto f_i$$

If $x \in U'$ we need to check that the map $\mathcal{L}|_{U'} \underset{\cong}{\xrightarrow{\varphi'}} \mathcal{O}_{U'}$ given by $s_i \mapsto f_i$ satisfies:



where the isomorphism γ is given by $s \in \mathcal{O}_X^*(U \cap U')$ and $f_i' = s \cdot f_i, s \in \mathcal{O}_X^*(U \cap U')$.

Remark 11.8. Although we cannot evaluate a section of $s \in \Gamma(X, \mathcal{L})$ at a closed point $x \in X$ to get an element of k, it does make sense to say $s(x) = 0$ or not.

Lecture 22. March 31, 2009

Let X be an integral scheme and $\mathcal{L}$ an invertible sheaf. Let $s \in \mathcal{L}(U)$. In what sense can we think of s as a regular function on U?

Example 11.9. $X = \mathbb{P}^n_A, \mathcal{L} = \mathcal{O}_x(1)$. $\Gamma(X, \mathcal{L})$ is spanned as an $\mathcal{O}_X(X) = A$-module by $x_0, \ldots, x_n$. Let $U_i = D_+(x_i)$. Then $\mathcal{O}_X(U_i) = A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]$ and

$$\mathcal{L}(U_i) = \left[A[x_0, \ldots, x_n]\left[\frac{1}{x_i}\right]\right]_1 = x_i \cdot \mathcal{O}_X(U_i) \xrightarrow{\cong} \mathcal{O}_X(U_i)$$

$x_i|_{U_i} \in \mathcal{L}(U_i)$ corresponds to $1 \in \mathcal{O}_X(U_i)$ whereas $x_i|_{U_j} \in \mathcal{L}(U_j)$ is $x_j \frac{x_i}{x_j} \in x_j \cdot \mathcal{O}_X(U_j)$ and corresponds to $\frac{x_i}{x_j} \in \mathcal{O}_X(U_j)$.

If U is sufficiently small, then

$$\mathcal{L}(U) \xrightarrow{\cong} \mathcal{O}_X(U) \quad \text{given by} \quad s \leftrightarrow f$$

Recall: for $f \in \mathcal{O}_X(U), p \in U$ then $f(p)$ means the image of f under the map $\mathcal{O}_{X,p} \longrightarrow \mathcal{O}_{X,p}/m_p = k(p)$.
s cannot be evaluated at a point $p \in U$, but we can think of $s(p)$ as the image of s in $\mathcal{L}_p \otimes \mathcal{O}_{X,p} k(p)$. However, it does make sense to look at the "zero set" of $s \in \Gamma(X, \mathcal{L})$ and in doing so we get a divisor on X corresponding to $s \in \Gamma(X, \mathcal{L})$. Choose a trivialization of $\mathcal{L}$: Cover $X = \cup_i U_i$ such that $\mathcal{L}|_{U_i} \xrightarrow{\cong} \mathcal{O}_X|_{U_i} = \mathcal{O}_{U_i}$. For $s \in \Gamma(X, \mathcal{L})$, let $s_i = s|_{U_i} \mapsto f_i \in \mathcal{O}_{U_i}$. The divisor of s is $\{(U_i, div(f_i))$ on $U_i \cap U_j$:

$$\begin{array}{ccccccc}
\mathcal{L}|_{U_i} & \xrightarrow{\varphi_i} & \mathcal{O}_{U_i} & \qquad & s_i = s|_{U_i} & \longrightarrow & f_i \\
\downarrow \text{rest.} & & \downarrow \text{rest.} & & \downarrow & & \downarrow \\
\mathcal{L}|_{U_i \cap U_j} & & \mathcal{O}_{U_i \cap U_j} & & s_i|_{U_j} = s_j|_{U_i} & & f_i \\
\uparrow & & \gamma \downarrow \cong & & \uparrow & & \gamma \downarrow \\
 & & \mathcal{O}_{U_i \cap U_j} & & & & s_{ij} f_i = f_j \\
\uparrow \text{rest.} & & \uparrow \text{rest.} & & \uparrow & & \uparrow \\
\mathcal{L}|_{U_j} & \xrightarrow{\varphi_j} & \mathcal{O}_{U_j} & & s_j = s|_{U_j} & \longrightarrow & f_j
\end{array}$$

where γ is given by multiplcation by a unit: $f_i \in \mathcal{O}_{U_i \cap U_j} \xrightarrow{\gamma} s_{ij} f_i = f_j$, $s_{ij} \mathcal{O}_X^*(U_i \cap U_j)$ is a unit. Given $s, t \in \Gamma(X, \mathcal{L})$, $\frac{s}{t}$ does make sense as an element of $\mathcal{O}_X(U_1 \cap U_2)$ where $U_i = X \setminus \{\text{zero set of } s_i\}$ is open. Evaluate $\frac{s}{t}$ at p:

$$\text{If } p \in U_i, \text{ then } \frac{s}{t} \xrightarrow{\varphi_i} \frac{f_i}{g_i} \quad \text{and} \quad \text{if } p \in U_j, \text{ then } \frac{s}{t} \xrightarrow{\varphi_j} \frac{f_j}{g_j} = \frac{s_{ij} f_i}{s_{ij} f_j} = \frac{f_i}{g_i}$$

So evaluating at p does not depend on our open patch.

Definition 11.10. Global section $s_0, \dots, s_n \in \Gamma(X, \mathcal{F})$ globally generate the coherent sheaf $\mathcal{F}$ if $\forall p \in X$ the germs $(s_0)_p, \dots, (s_n)_p \in \mathcal{F}_p$ generate $\mathcal{F}_p$ as an $\mathcal{O}_{X,p}$-module. If $\mathcal{F}$ is invertible, then $s \in \Gamma(X, \mathcal{F})$ generates $\mathcal{F}$ at $p \Leftrightarrow s_p \in \mathcal{F}_p$ generates as an $\mathcal{O}_{X,p}$-module $\Leftrightarrow s_p \notin m_p\mathcal{F}_p, m_p \subseteq \mathcal{O}_{X,p} \Leftrightarrow s(p) \neq 0$, where $s(p)$ is the image of s in $\mathcal{F} \otimes k(p) = \mathcal{F}_p/(m_p\mathcal{F}_p)$.

Example 11.11. $X = \mathbb{P}^n_A$:

- $\mathcal{L} = \mathcal{O}_X(1)$: the sections $x_0, \dots, x_n$ are the global generators for $\mathcal{L}$
- $\mathcal{L} = \mathcal{O}_X(t)$: $x_0^t, \dots, x_n^t$ are global generators
- In general, $s_0, \dots, x)n \in \Gamma(X, \mathcal{L})$ with $\mathcal{L}$ any invertible sheaf on X any integral scheme. These s_i generate $\mathcal{L} \Leftrightarrow \cup X_i = X$ where $X_i = X \setminus \{\text{zero set of } s_i\}$.

Example 11.12. Let $X = \text{Proj}\left(\frac{k[x,y,z]}{(z^2-xy)}\right) = D_+(x) \cup D_+(y)$ and $S = \frac{k[x,y,z]}{(z^2-xy)}$.
$\Gamma(X, \mathcal{O}_X(1)) = \mathcal{O}_X(1)(X) =$ degree 1 elements in S
$\mathcal{O}_X(1)(D_+(x)) = \left[S\left[\frac{1}{x}\right]\right]_1 = x \cdot [S[\frac{1}{x}]]_0 = x \cdot k\left[\frac{z}{x}\right]$ is a module over

$$\mathcal{O}_X(D_+(x)) = \left[S\left[\frac{1}{x}\right]\right]_0 = \left[\frac{k[x,y,z]}{(z^2-xy)}\left[\frac{1}{x}\right]\right]_0 = \frac{k[\frac{y}{x}, \frac{z}{x}]}{(\frac{z}{x})^2 - (\frac{y}{x})} \cong k\left[\frac{z}{x}\right]$$

$\Gamma(X, \mathcal{O}_X(1))$ is globally generated by x, y.
Note that: given $X \overset{\varphi}{\hookrightarrow} \mathbb{P}^2_k$ then $\varphi^*\mathcal{O}_{\mathbb{P}^n_A}(1) = \mathcal{O}_X(1)$.

Example 11.13. Given any morphism of schemes over A, $X \xrightarrow{\varphi} \mathbb{P}^n_A$, $\mathcal{L} = \varphi^*\mathcal{O}_{\mathbb{P}^n_A}(1)$ is an invertible sheaf which is globally generated by $s_i = \varphi^* x_i, \forall i = 0, \dots, n$, where $x_i \in \Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(1))$.
Say $\mathcal{O}_{\mathbb{P}^n_A}(1)$ is trivialized on $U_i = D_+(x_i)$. $\mathcal{O}_{\mathbb{P}^n_A}(1) \xrightarrow{\cong} \mathcal{O}_{\mathbb{P}^n_A}(U_i)$. Because U_i cover $\mathbb{P}^n_A$, $X_i = \varphi^{-1}(U_i)$ is an open cover of X.

$$\varphi^*\mathcal{O}_{\mathbb{P}^n_A}(1)(x_i) = \mathcal{O}_X(x_i) \otimes_{\mathcal{O}_{p^n_A}} \mathcal{O}_{\mathbb{P}^n_A}(1)(U_i) \underset{\cong}{\xrightarrow{1\otimes\varphi_i}} \mathcal{O}_X(x_i) \otimes_{\mathcal{O}_{\mathbb{P}^n_A}} \mathcal{O}_{\mathbb{P}^n_A}(U_i)$$

Amazing Fact:

Theorem 11.14. *Let X be any scheme over A.*

1. *If $X \xrightarrow{\varphi} \mathbb{P}^n_A$ is a morphism of A-schemes, then $\mathcal{L} = \varphi^*\mathcal{O}_{\mathbb{P}}(1)$ is an invertible sheaf globally generated by $s_0, \dots, s_n$, where $s_i = \varphi^*(x_i)$.*
2. *Conversely, if $\mathcal{L}$ is an invertible sheaf on X globally generated by $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ then there is a unique morphism of A-schemes $X \xrightarrow{\varphi} \mathbb{P}^n_A$ such that $\varphi^*\mathcal{O}_{\mathbb{P}}(1) = \mathcal{L}$ and $\varphi^* x_i = s_i$.*

Proof. This is theorem 7.1 in Hartshorne II.

1. Already proven
2. Given $X, \mathcal{L}$ and $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Let $X_i = X - \{\text{zero set of } s_i\}$ open. Since s_i globally generate $\mathcal{L}$, the X_i cover X. Define:
$$X_i \xrightarrow{\varphi_i} U_i = \text{Spec } A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \subseteq \mathbb{P}^n_A, U_i = D_+(x_i)$$
in such a way that it will be obvious that $\varphi_i|_{X_i \cap X_j} = \varphi_j|_{U_i \cap U_j}$ so they glue to a morphism $X \longrightarrow \mathbb{P}^n_A$. Because U_i is affine, to construct φ_i, it is equivalent to construct an A-algebra map
$$A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \longrightarrow \mathcal{O}_{X_i}(X_i) \text{ given by } \frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}$$
It is easy to check that these patch together to define $\varphi : X \longrightarrow \mathbb{P}^n_A$ with the desired properties.

□

Example 11.15. $X = \text{Proj}\left(\frac{k[x,y,z]}{(z^2-xy)}\right)$, $\Gamma(X, \mathcal{O}_X(1)) = k$-span of x, y, z.
The map which corresponds to the generators x, y, z is:

$$X \longrightarrow \mathbb{P}^2 \text{ given by } [\lambda_0 : \lambda_1 : \lambda_2] \mapsto [\lambda_0 : \lambda_1 : \lambda_2]$$

$\varphi^*(x) = \overline{x}, \varphi^*(y) = \overline{y}, \varphi^*(z) = \overline{z}$ Note also that x, y or $x, x+y$ are also global generators.
The map which corresponds to the generators x, y is:

$$X \longrightarrow \mathbb{P}^1 \text{ given by } [\lambda_0 : \lambda_1 : \lambda_2] \mapsto [\lambda_0 : \lambda_1], \text{ where } \lambda_0\lambda_1 = \lambda_2^2$$

Example 11.16. What are all of the k-automorphisms of $\mathbb{P}^n_k$?

$$Aut_k(\mathbb{P}^n_k) = \{\varphi : \mathbb{P}^n_k \longrightarrow \mathbb{P}^n_k \mid \varphi \text{ is an isomorphism over } k\}$$

Map of groups:

$$0 \longrightarrow k^* \cdot Id \longrightarrow GL_{n+1}(k) \xrightarrow{\alpha} Aut(\mathbb{P}^n_k) \text{ where } \alpha(A) = (\text{corresponding linear transformation})$$

$$\frac{GL_{n+1}(k)}{k \cdot I} = PGL_n(k) \hookrightarrow Aut(\mathbb{P}^n_k)$$

Claim: This is an isomorphism of groups
Given an automorphism $\mathbb{P}^n \xrightarrow{\varphi} \mathbb{P}^n$, $\varphi^*(\mathcal{O}_1) = \mathcal{L} \in Pic(\mathbb{P}^n) \cong \mathbb{Z}$. Since $Pic(\mathbb{P}^n) \xrightarrow{\varphi^*} Pic(\mathbb{P}^n)$ is an isomorphism of groups, $\varphi^*\mathcal{O}(1)$ must be a generator of $Pic(\mathbb{P}^n)$, i.e. $\mathcal{L}$ is either $\mathcal{O}(1)$ or $\mathcal{O}(-1)$. But since $\Gamma(\mathbb{P}^n, \mathcal{O}(-1)) = 0, \mathcal{L} = \mathcal{O}(1)$.
$\varphi^* x_i = s_i \in \Gamma(\mathbb{P}^n, \mathcal{O}(1)) = \mathcal{O}(1)(\mathbb{P}^n) = \mathcal{L}(\mathbb{P}^n) = [k[x_0, \ldots, x_n]]_1$ and $s_i = \sum_{j=0}^n a_{ij} x_j$. Also, (a_{ij}) is a full rank matrix since s_i generate.

Lecture 23. April 2, 2009

Exercise 11.17. Hartshorne II: 7.1,7.2,7.3,7.5 due on Thursday, April 9.

$$\{X \xrightarrow{\varphi} \mathbb{P}^n_A \text{ morphisms of } A\text{-schemes}\} \leftrightarrow \{\mathcal{L} \text{ invertible sheaf on } X \text{ with generators } s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})\}$$

$$(X \xrightarrow{\varphi} \mathbb{P}^n_A) \longrightarrow (\mathcal{L} = \varphi^*\mathcal{O}(1), s_i = \varphi^* x_i)$$

$$\text{``}\left(X \longrightarrow \mathbb{P}^n_A, x \mapsto [s_0(x) : \cdots : s_n(x)]\right)\text{''} \longleftarrow \left(\mathcal{L}, s_0, \ldots, s_n \in \Gamma(X, \mathcal{L}) \text{ which generate}\right)$$

Literally: $X \xrightarrow{\varphi} \mathbb{P}^n_A$ is obtained by glueing the maps φ_i corresponding to the ring maps

$$\Gamma(X_i, \mathcal{O}_X) \longleftarrow A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]$$

$$\frac{s_j}{s_i} \longleftarrow \frac{x_j}{x_i}$$

where $x_i = \{p \in X \mid s_i(p) \neq 0\} = \{p \in X \mid s_i \notin m_p\mathcal{L}_p\} \subseteq X$ open. This tells us how to interpret $s_i(x)$.

Question: *What if, instead of $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$, ($A \longrightarrow \mathcal{O}_X(X)$-module) we choose a different set of generators $t_0, \ldots, t_n$ of A-module generators for the A-submodule of $\Gamma(X, \mathcal{L})$ generated by $s_0, \ldots, s_n$?*
Write $s_i = \sum_{j=0}^n a_{ij} t_j$ and $t_i = \sum_{j=0}^n b_{ij} s_j$ so that there is an invertible matrix that changes the generators. The maps $\varphi_s : X \longrightarrow \mathbb{P}^n_A$ and $\varphi_t : X \longrightarrow \mathbb{P}^n_A$ correspond to the maps $x \mapsto [s_0(x) : \cdots : s_n(x)]$ and $x \mapsto [t_0 : \cdots : t_n]$, respectively.
Answer: The corresponding maps φ_s and φ_t differ by an automorphism of $\mathbb{P}^n_A$
Question: *What if $\mathcal{L}$ is invertible, $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$, but the s_i do not generate $\mathcal{L}$?*

Let $U = X - \mathbb{V}(s_0, \dots, s_n)$, $\mathcal{L}|_U$ is invertible on $U \subseteq X$ open and $s_i|_U \in \Gamma(U, \mathcal{L}|_u)$ globally generate $\mathcal{L}|_U$. Then instead of the map $X \xrightarrow{\varphi} \mathbb{P}^n$ we have a map $U \longrightarrow \mathbb{P}^n$.

Definition 11.18. $\mathcal{L}$ an invertible sheaf is **very ample** if there exist $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ that generate $\mathcal{L}$ and the corresponding map $X \xrightarrow{\varphi} \mathbb{P}^n_A$ given by $x \mapsto [s_0(x) : \cdots : s_n(x)]$ is a closed immersion (embedding).

Definition 11.19. $\mathcal{L}$ is **ample** means that $\mathcal{L}^m$ is very ample for some $m > 0$.

Remark 11.20. This is not the way that Hartshorne defines ample, allthough they are often equivalent, however this is the definition that most mathematicians use.

Definition 11.21. A **variety** X over $k = \overline{k}$ is an integral, separated scheme of finite type over k.

X has a cover by open sets U where $U = \text{Spec}\left(\frac{k[x_0, \dots, x_n]}{(g_1, \dots, g_t)}\right)$. Since X is integral, $\frac{k[x_0, \dots, x_n]}{(g_1, \dots, g_t)}$ is a domain and since X is of finite type, $\frac{k[x_0, \dots, x_n]}{(g_1, \dots, g_t)}$ has finitely many generators and finitely many relations.

Definition 11.22. A **curve** is a variety of dimension 1.

Often when someone says "curve" they mean a smooth variety of dimension 1, projective.

Definition 11.23. A **surface** is a variety of dimension 2.

Definition 11.24. A **smooth variety** (or **non-singular**) is a variety X over k where $\Omega_{X/k}$ is a locally free $\mathcal{O}_X$-module of rank equal to the dimension of X and all local rings $\mathcal{O}_{X,p}$ are regular (i.e. $\dim \mathcal{O}_{X,p} = \dim m_p/m_p^2$).

Remark 11.25. Smooth varieties and non-singular varieties are not the same, however they are when we assume that they are defined over $k = \overline{k}$.

Definition 11.26. A **complete variety** (or **proper**) X is a variety such that $X \longrightarrow \text{Spec } k$ is a proper map.

Big Questions About Varieties:

- How can we tell if two varieties are isomorphic? Or birationally equivalent?
- Given a birational equivalence class of varieties can we choose a "canonical representative"? *(Minimal Model program)*
- How can we construct Moduli spaces of complete, smooth curves of genus g? *($\mathcal{M}_g$)*
- How can we construct Moduli spaces of surfaces or higher-dimensional varieties? *(open question)*
- What is the smallest dimensional projective space into which a variety X can be embedded? *(smooth surfaces always embed into $\mathbb{P}^5$ but for any number m there is a singular variety that does not embed into $\mathbb{P}^m$)*
- Given a smooth projective curve X of genus g, how can we view X as a branched cover (i.e. ramified cover) of $\mathbb{P}^1$?

Tools:

(1) Need criteria to check wheter a given $\mathcal{L}$ on X is very ample
(2) Given an invertible sheaf $\mathcal{L}$ on a projective variety over k, need formulas for $\dim\big(\Gamma(X, \mathcal{L})\big) = \dim\big(H^0(X, \mathcal{L})\big)$? *Riemann Roch! Vanishing theorems.*

Proposition 11.27. *Let X be a projective scheme over k and $X \xrightarrow{\varphi} \mathbb{P}^n_k$ a morphism given by $\mathcal{L}$ and $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Let $V \subseteq$ Gamma$(X, \mathcal{L})$ be the vector space generated by the $\{s_0, \dots, s_n\}$. The morpism φ is an embedding (i.e. $\mathcal{L}$ is very ample) If and only if the following conditions hold:*

(1) *$\forall p,q \in X, \exists s \in V$ such that $s(p)=0$ but $s(q)\neq 0$ (or vice versa). (i.e. $\mathcal{L}$ "separates points")*
(2) *$\forall p \in X, S=\{s \in V \mid s(p)=0\}$ the germs (s_p) of these $s \in S$ in $\mathcal{L}_p$ span $m_p/m_p^2 \otimes \mathcal{L} \subseteq \mathcal{L}_p/m_p\mathcal{L}_p$ (i.e. $\mathcal{L}$ "separates tangent vectors")*

Condition one is equivalent to saying $X \xrightarrow{\varphi} \mathbb{P}^n$ is one-to-one onto its image:
For $p,q \in X$ assume that $s_0(p)=0, s_0(q)\neq 0$ (up to automorpism of $\mathbb{P}^n$, s_0 is the s from condition 1) so that $\varphi(p)=[0 : s_1(p) : \cdots : s_n(p)]$ and $\varphi(q)=[1 : \frac{s_1}{s_0}(q) : \cdots : \frac{s_n}{s_0}(q)]$ $\Rightarrow \varphi(p)\neq\varphi(q)$.

11.1. **Curves:** Hartshorne IV
Let X be a smooth projective curve over $k=\overline{k}$. $\omega_X=\Omega_X$ is locally free of rank 1 (i.e. invertible). Canonical sheaf.

$$Pic(X)=\{\text{Invertible sheafs}\} \leftrightarrow \frac{Div(X)}{P(X)}=Cl(X)$$
$$\omega_X \leftrightarrow \text{ divisor class } K_X$$

Definition 11.28. The **genus** of X is $\dim\big(\Gamma(X,\omega_X)\big)$.

Definition 11.29. Let D be a divisor on X, $D=\sum n_i p_i$, where p_i are points and $n_i \in \mathbb{Z}$. The **degree** of D is $\sum n_i$.

Proposition 11.30. *The group map*

$$Div(X) \longrightarrow \mathbb{Z} \text{ given by } D=\sum n_i p_i \mapsto deg(D)=\sum n_i$$

has $P(X)$ in the kernel, i.e. it determines a group map $Cl(X) \xrightarrow{degree} \mathbb{Z}$.

Due to this, we can define:

Definition 11.31. The **degree** of an invertible sheaf $\mathcal{L}$ on a curve X is $deg(\mathcal{L})=deg(D)$, where D is any divisor such that $\mathcal{L}\cong\mathcal{O}_X(D)$ (D is uniquely determined up to a principal divisor).

Observation: If $\mathcal{L}$ has a global section, $deg(\mathcal{L})\geq 0$.
Easy criterion: $\mathcal{L}$ is ample $\Leftrightarrow deg(\mathcal{L})>0$.

Lecture 24. April 7, 2009

Let X be a variety (Classical case: *a quasi-projective variety* or for *Schemes:* an integral, separated scheme of finite type over $k=\overline{k}$).

Definition 11.32. $X \overset{\varphi}{\dashrightarrow} Y$ is a **rational map** if it is a morphism defined on a dense open subset of X. It is an equivalence class of such morphisms.

If $U \xrightarrow{\varphi_U} Y$ and $V \xrightarrow{\varphi_V} Y$, where $U,V \subseteq X$ are dense open sets, then $\varphi_U \sim \varphi_V$ if $\varphi_U|_{U\cap V}=\varphi_V|_{U\cap V}$.

Definition 11.33. $X \overset{\varphi}{\dashrightarrow} Y$ is regular at $p\in X$ if $\exists p \in U \subseteq X$ such that φ is represented by a morphism $U \xrightarrow{\varphi} Y$ on a dense open subset $U\subset X$.

Definition 11.34. The **locus of indeterminacy** of $X \overset{\varphi}{\dashrightarrow} Y$ is the closed set

$$\{p \in X \mid \varphi \text{ is not regular at } p\}.$$

Theorem 11.35. *If $X \overset{\varphi}{\dashrightarrow} Y$ is a rational map with X normal and Y projective, then the locus of indeterminacy has codimension atleast 2.*

In particular, if X is a curve (smooth), then $X \overset{\varphi}{\dashrightarrow} Y$ is actually a morphism.

Definition 11.36. Two varieties are **birationally equivalent** if any of the following hold:

(1) $\exists X \overset{\varphi}{\dashrightarrow} Y$ and $Y \overset{\psi}{\dashrightarrow} X$ which are mutually inverse (as rational maps)
(2) $\exists U \subseteq X, V \subseteq Y$ dense open sets with $U \xrightarrow{\cong} V$
(3) $k(X) \cong_k k(Y)$

If X and Y are smooth projective curves, then X is birationally equivalent to $Y \Leftrightarrow X \cong Y$. For this lecture and Hartshorne IV, a curve is a "smooth projective variety of dimension 1." Let X be a curve with divisor $D = \sum_{i=1}^{t} n_i P_i \in DivX$ (P_i are points) whose degree is $\sum_{i=1}^{t} n_i \in \mathbb{Z}$. The map

$$Div(X) \longrightarrow \mathbb{Z}, \text{ given by } D \mapsto deg(D), \text{ is a group homomorphism.}$$

Proposition 11.37. *This map induces a group homomorphism $Cl(X) \longrightarrow \mathbb{Z}$. Equivalently, $deg(div(f)) = 0, \forall f \in k(X)$.*

A morphism $X \overset{\varphi}{\twoheadrightarrow} Y$ of curves induces a map of fields $k(Y) \hookrightarrow k(X)$ as follows:
The map $\mathcal{O}_Y(U) \longrightarrow \varphi_*\mathcal{O}_X(U)$ is injective $\forall U \subseteq Y$ so there is an inclusion of domains

$$\mathcal{O}_Y(U) \hookrightarrow \mathcal{O}_X(\varphi^{-1}(U)) \overset{rest.}{\hookrightarrow} \mathcal{O}_X(V), \textit{ where } V \subseteq \varphi^{-1}(U) \textit{ is open.}$$

Since $\mathcal{O}_Y(U) \subseteq k(Y)$ and $\mathcal{O}_X(V) \subseteq k(X)$, there is an inclusion of fields $k(Y) \hookrightarrow k(X)$.
$k(X), k(Y)$ have transcendence degree 1 over k and the induced map $k(Y) \hookrightarrow k(X)$ is a finite algebraic extension of fields. Define the degree of φ as:

$$deg(\varphi) = \text{ the degree of the field extension } [k(Y) : k(X)]$$

Proposition 11.38. *Let $X \overset{\varphi}{\twoheadrightarrow} Y$ be a morphism of curves and $D \in DivY$. Then*

$$deg(\varphi^*(D)) = deg(\varphi) \cdot deg(D).$$

Proof. Let $D = \sum n_i P_i$, then $\varphi^*(D) = \sum n_i \varphi^*(P_i)$ (i.e. $\varphi^*(P_i)$ is the preimage of P_i under φ counted with multiplicities) $\Rightarrow deg(\varphi^*(D)) = \sum n_i deg(\varphi^* P_i) = (\sum n_i)(deg(\varphi)) = deg(D)deg(\varphi)$. It suffices to show that $deg(\varphi^*(P) = deg(\varphi)$.
$X \overset{\varphi}{\twoheadrightarrow} Y$ and $U \subseteq X, V \subseteq Y$ with $V \xrightarrow{\varphi} U$ and $P \in U$. $\mathcal{O}_{Y,P}$ is a DVR so let t be the generator of the maximal ideal m_p in $\mathcal{O}_{Y,P}$. Then $\varphi^* t \in \mathcal{O}_X(V)$. $\mathcal{O}_X(V) \otimes \mathcal{O}_{Y,P}$ is a DVR. By definition $\varphi^{-1}(P)$ is the subscheme of X defined by $\varphi^*(t)$ in V. □

Example 11.39. Let $S = \frac{k[x,y,z]}{xz-y^2}$ and $X = \text{Proj } S \xrightarrow{\varphi} Y = \text{Proj } k[x,z] = \mathbb{P}^1_k$ where $[a : b : c] \mapsto [a : c]$. The map φ is given by $x, z \in \Gamma(X, \mathcal{O}_X(1))$.
In Y: $D_+(z) = \text{Spec } k[\frac{x}{z}] = \text{Spec } k[t]$
In X: $D_+(z) = \text{Spec } \frac{k[\frac{x}{z}, \frac{y}{z}]}{\frac{x}{z} - (\frac{y}{z})^2} = \text{Spec } \frac{k[t,s]}{t-s^2}$,
where $t = \frac{x}{z}, s = \frac{y}{z}$.

$$k[t] \oplus s \cdot k[t] \cong k[t^{\frac{1}{2}}] \cong k[s] \cong \frac{k[t,s]}{t-s^2} \longleftarrow k[t] \quad \Rightarrow k(t^{\frac{1}{2}}) \cong k(s) \hookleftarrow k(t),$$

which has degree 2. Let $P = [\lambda : \mu]$, wlog $\mu \neq 0$ so instead write $P = [\lambda : 1]$.
$P \in D_+(z) = \text{Spec } k\left[\frac{x}{z}\right] = \text{Spec } k[t] = \mathbb{A}^1 \Leftrightarrow$ local parameter for P is $(t - \lambda)$
$\varphi^* P =$ preimage of P, counting multiplicities.
$\varphi^{-1}(P) =$ subscheme of $D_+(z) = \text{Spec } \frac{k[t,s]}{t-s^2} \subseteq X$ defined by $\varphi^*(t - \lambda)$. So $\varphi^{-1}(P)$ is the

subscheme of Spec $\frac{k[t,s]}{t-s^2}$ defined by $(t-\lambda)=(s^2-\lambda)$.

$$\varphi^{-1}P = \text{Spec Spec } \frac{k[t,s]}{(t-s^2, t-\lambda)} \cong \text{Spec } \frac{k[s]}{s^2-\lambda}$$

This has two points $(\lambda, \pm\sqrt{\lambda})$ or a point $(0,0)$ with multiplicity 2. So $\varphi^*P = P_1 + P_2 = [\lambda : \sqrt{\lambda} : 1] + [\lambda : -\sqrt{\lambda} : 1]$ (note if $\lambda = 0$ then this becomes $2[0:0:1]$). Therefore $deg(\varphi^*P) = 2$.

Of Prop. Take $f \in k(X)$ and define $X \overset{\varphi}{\dashrightarrow} \mathbb{P}^1$ by $x \mapsto [f(x):1]$. Note that if f has a pole at $x \in X$, then $\varphi(x) = [1:0]$.

$$div(f) = \text{zeros of } f - \text{poles of } f = \varphi^*([0:1]) - \varphi^*([1:0]) = \varphi^*([0:1]-[1:0])$$

$$deg(div(f)) = deg(\varphi^*([0:1]-[1:0])) = deg(\varphi)\cdot deg([0:1]-[1:0]) = deg(\varphi)\cdot 0 = 0$$

So the degree of $div(f)$ is zero. □

Tools for Studying Curves:

(1) **Serre Duality for curves:**
Let X be a smooth projective curve ($\omega_X = \Omega_X$ is an invertible sheaf). Then $\Gamma(X,\mathcal{L})$ is dual to $H^1(X, \omega_X \otimes \mathcal{L}^{-1})$.
Equivalently, let $\mathcal{L} = \mathcal{O}_X(D), \omega_X = \mathcal{O}_X(K_X)$, then $\Gamma(D)$ is dual to $H^1(K_X - D)$.

(2) **Riemann Roch formula:**
Let X be a smooth projective curve and $\mathcal{L}$ be an invertible sheaf. The genus of X, $g(X)$, is $g(X) = dim_k(\Gamma(X,\omega_X)) = dim_k(H^1(X,\mathcal{O}_X))$. Then

$$dim(\Gamma(X,\mathcal{L})) = deg(\mathcal{L}) + 1 - g + dim(\Gamma(X, \mathcal{L}^{-1}\otimes\omega_X))$$

Remark 11.40. Riemann's formula was $dim(\Gamma(X,\mathcal{L})) \geq deg(\mathcal{L}) + 1 - g$ and Roch found the error term $dim(\Gamma(X,\mathcal{L}^{-1}\otimes\omega_X))$, which can be annoying to compute but is certainly non-negative.

What is $deg(\omega_X)$?
$dim(\Gamma(X,\omega_X)) = deg(\omega_X) + 1 - g + dim(\Gamma(X,\Omega_X)), dim(\Gamma(X,\Omega_X)) = 1, dim(\Gamma(X,\omega_X)) = 1 \Rightarrow g = deg(\omega_X) + 1 - g + 1 \Rightarrow deg(\omega_X) = 2g-2$.
Fact: If $degD < 0$, then $\Gamma(X, \mathcal{O}(D)) = 0$.

Proof. If $s \in \Gamma(X, \mathcal{O}_X(D))$, then the zero set of s is an effective divisor D' linearly equivalent to D and so $degD = degD' \geq 0$. □

Example 11.41. $dim(\Gamma(X,\mathcal{L}^{-1})\otimes\omega_X) = \dim(\Gamma(X, K_X - D)) = 0$ if $deg(K_X - D) < 0$.

Proposition 11.42. *If $degD > 2g-2$, then $dim(\Gamma(D)) = degD + 1 - g$.*

Lecture 25. April 9, 2009

For this lecture a curve is always a smooth, projective variety of dimension 1 over $k = \overline{k}$.

Warm-up Problem: What can we say about a curve X if $\exists p, q \in X$ such that $p \neq q$ but $p \sim q$ (i.e. $p - q = div(f), f \in k(X)^*$)?
If this were true for all points of X, then $Cl(X) \cong \mathbb{Z}$ since any point represents the same class and 2 times any point represents the same class as 2 times any other point. In general,

$$X \xrightarrow{\varphi} \mathbb{P}^1 \text{ given by } z \mapsto [f(x):1]$$

φ is surjective since f is not a constant and $f \in k(X)^*$. Also $p - q = div(f)$, f has a simple zero at p and a simple pole at q. Therefore $deg(\varphi) = deg(\varphi^*([0:1]) = deg(p) = 1$ $\Rightarrow k(X)\mathcal{L} \overset{\cong}{\hookrightarrow} k(t)$ since X is a curve $\Rightarrow X$ is birationally equivalent to $\mathbb{P}^1$. Hence φ is an isomorphism.

Recall: $deg(\mathcal{L})$ = the degree of any divisor representing $\mathcal{L}$

$$H^0(X,\mathcal{L}) \longrightarrow \{\text{effective divisors on } X \text{ all linearly equivalent}\}$$

$$s \mapsto (\text{divisors of zeros of } s$$

$$\lambda s \mapsto \{s=0\} \Leftrightarrow \{\lambda s = 0\}$$

Therefore $\mathbb{P}(H^0(X,\mathcal{L})) = \{\text{effective divisors all linearly equivalent}\}$.
genus$(X) = h^0(X,\omega_X) = dim(H^0(X,\omega_X) = h^1(X,\mathcal{O}_X)$.

Remark 11.43. h^0 is the dimension of H^0 and ω_X is the top dimensional forms on X.

Serre Duality: $H^0(X,\mathcal{L})$ is dual to $H^1(X,\omega_X \otimes \mathcal{L}^{-1})$
$deg(\omega_X) = 2g-2$.

Remark 11.44. The genus defined here agrees with the definition in topology. In particular, the genus in topology is $\frac{1}{2}dim_{\mathbb{R}}H^1_{Sing.}(X,\mathbb{R})) = dim_{\mathbb{C}}H^1_{Sing.}(X,\mathbb{C}))$. On a smooth projective variety X, viewing X with its complex topology instead of its Zariski topology and using hodge decomposition:

$$H^n_{Sing.}(X,\mathbb{C}) \cong \oplus_{p+q=n}H^{pq}(X) = \oplus_{p+q=n}H^p(X,\wedge^q\Omega_X)$$

For curves,

$$H^1_{Sing.}(X,\mathbb{C}) = H^0(X(,\omega_X) \oplus H^1(X,\mathcal{O}_X)$$

And $dim_{\mathbb{C}}H^1_{Sing.}(X,\mathbb{C}) = 2g$.

Rieman Roch *(RR):* Let X be a curve and $\mathcal{L}$ an invertible sheaf.

$$h^0(X,\mathcal{L}) = deg(\mathcal{L}) + 1 - g + h^0(\omega_X \otimes \mathcal{L}^{-1}),$$

often the last term is zero. In particular this happens if $deg(\omega_X \otimes \mathcal{L}^{-1} < 0$. Since $deg(\omega_X \otimes \mathcal{L}^{-1}) = deg(\omega_X) - deg(\mathcal{L}) = 2g-2$, this amounts to $g < 1$.

Proof. First of all, $h^0(\omega_X \otimes \mathcal{L}^{-1}) = h^1(X,\mathcal{L})$, $h^0(\mathcal{O}_X) = 1$, and $g = h^1(\mathcal{O}_X)$. So we can arrange the equation to become:

$$\chi(\mathcal{L}) \equiv h^0(X,\mathcal{L}) - h^1(X,\mathcal{L} = deg(\mathcal{L} + h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X)$$

Let $\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X)$ and $\mathcal{L} = \mathcal{O}_X(D)$. Fix $p \overset{i}{\hookrightarrow} X$. Then $\mathcal{O}_X \longrightarrow i_*\mathcal{O}_p$ by $f \mapsto f(p)$.

$$0 \longrightarrow \mathcal{O}_X(-p) \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_p \longrightarrow 0$$

$i_*\mathcal{O}_p$ is the skyscraper sheaf k at p, $\mathcal{O}_X(-p) = \{g \in k(X) \mid div(g) - p \geq 0\}$.
Apply $\otimes_{\mathcal{O}_X}\mathcal{O}_X(D)$ to get the exact sequence:

$$0 \to \mathcal{O}_X(-p)\otimes_{\mathcal{O}_X}\mathcal{O}_X(D) = \mathcal{O}_X(D-p) \to \mathcal{O}_X\otimes_{\mathcal{O}_X}\mathcal{O}_X(D) = \mathcal{O}_X(D) \to i_*\mathcal{O}_p\otimes_{\mathcal{O}_X}\mathcal{O}_X(D) \to 0$$

$i_*\mathcal{O}_p \otimes_{\mathcal{O}_X} \mathcal{O}_X(D) \simeq \mathcal{O}_p$ is still a skyscraper sheaf k at p.
Then we get a long exact sequence in cohomology:

$$0 \to H^0(X,D-p) \to H^0(X,D) \to H^0(p,\mathcal{O}_p) \to H^1(X,D-p) \to H^1(X,D) \to H^1(p,\mathcal{O}_p) = 0$$

Note that we are abusing notation so $H^0(X,D-p)$ is actually $H^0(X,\mathcal{O}_X(D-p))$. The alternating sum of these dimensions is zero since the sequence is exact So:

$$h^0(D-p) - h^0(D) + h^0(p,\mathcal{O}_p) - h^1(D-p) + h^1(D) + 0 = 0$$

Noting that $h^0(p,\mathcal{O}_p) = 1$ and rearranging we get:

$$h^0(D-p) - h^1(D-p) = h^0(D) - h^1(D) - 1 \Rightarrow \chi(D-p) = \chi(D) - 1$$

RR holds for $D \Leftrightarrow$ RR holds for $D - p$

$\Leftrightarrow$ RR holds for $\chi(D-p) + deg(D-p) + \chi(\mathcal{O}_X)$

$\Leftrightarrow$ RR holds for $\chi(D-p) + deg(D) - 1 + \chi(\mathcal{O}_X)$

RR holds for $D \Leftrightarrow$ RR holds for $\chi(D) = deg(D) + \chi(\mathcal{O}_X)$

$\Leftrightarrow$ RR holds for $\chi(D-p) + deg(D) - 1 + \chi(\mathcal{O}_X)$ since $\chi(D-p) = \chi(D) - 1$

Therefore RR holds for all divisors if it holds for some divisor, so we only need to check this for one divisor, D. Try for $D = 0 : \chi(\mathcal{O}_X) = deg(\mathcal{O}_X) + \chi(\mathcal{O}_X)$ and $deg(\mathcal{O}_X) = 0$. □

What can be said about a curve of genus 0?

Theorem 11.45. *If X is a curve of genus 0, then $X \cong \mathbb{P}^1$.*

Proof. By RR, curves of genus 0 satisfy: $h^0(D) = deg(D)+1+h^0(K_X-D)$. For effective D, the degree of $K_X - D$ is $2g-2-deg(D) < 0$, so $h^0(K_X - D) = 0$. So $h^0(D) = deg(D)+1$. Take $D = p \Rightarrow h^0(D) = 2$. Fix $s_0, s_1 \in \Gamma(X, \mathcal{O}_X(D))$ that span.

$$X \xrightarrow{\varphi} \mathbb{P}^1 \text{ given by } x \mapsto [s_0(x) : s_1(x)]$$

$$deg(\varphi) = deg(\varphi^*([0:1]) = deg(\text{of the zero set, counting mult. of } s_0) = 1$$

Therefore $X \cong \mathbb{P}^1$. □

Hurewitz Formula: If $X \xrightarrow{f} Y$ is a map of curves with $deg(f) = n$ and $char(k) = 0$, then

$$2(\text{genus of X}) - 2 = n(2(\text{genus of Y}) - 2) + \sum_{p \in X} (e_p - 1),$$

where e_p is the multiplicity of p in $\varphi^*(\varphi(p)) = \varphi^{-1}(\varphi(p))$ with multiplicity. Note that $e_p = 1 \Leftrightarrow p$ is not a ramification point. For shorthand, g_X is the genus of X. *What can be said about a curve of genus 1?* Let X be a curve of genus 1 and $D > 0$ effective divisor on X. By RR

$$h^0(D) = deg(D) + 1 - g + h^0(K_X - D) = deg(D)$$

The degree of $K_X - D = 2g - 2 - degD = -degD < 0$, so $h^0(K_X - D) = 0$. Try $D = 2p$. Then $H^0(\mathcal{O}_X(2p))$ has dimension 2 and is spanned by s_0, s_1. Assume that $\{s_0 = 0\} = 2p$. Then we get a map:

$$X \xrightarrow{\varphi_{2p}} \mathbb{P}^1 \text{ given by } x \mapsto [s_0(x) : s_1(x)]$$

φ_{2p} cannot be a map to a point since if it were we would have $s_0(x) = s_1(x), \forall x \in X$, in which case s_0, s_1 are linearlly dependent and so they would not span. For now, let's assume they have no common factor (we will discuss this more carefully later). Then $deg(\varphi_{2p}) = 2$. By the Hurwitz formula, $2g_X - 2 = 2(2g_{\mathbb{P}^1} - 2) + \sum_{p \in X}(e_p - 1) \Rightarrow 0 = -4 + \sum_{p\in X}(e_p - 1) \Rightarrow \sum_{p \in X}(e_p - 1) = 4$.

Lecture 26. April 14, 2009

Let X be a smooth projective variety over $k = \overline{k}$. A morphism $x \xrightarrow{\varphi} \mathbb{P}^n_k$ is equivalent to a choice of an invertible sheaf $\mathcal{L}$ of $\mathcal{O}_X$-modules and global sections $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L}) = H^0(X, \mathcal{L})$ which globally generate $\mathcal{L}$.
Suppose now that X is a (smooth projective) curve. Take $\mathcal{L}$ with generators $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$

Example 11.46. Let $X = \mathbb{P}^1 = \text{Proj } k[s,t]$ and $\mathcal{L} = \mathcal{O}_X(3)$. Then $\Gamma(X, \mathcal{L}) =$vector space $[k[s,t]]_3$. Let $s_0 = s^2t, s_1 = st^2, s_2 = t^3$. Where do these generate $\mathcal{L}$?
Let $p = [0:1] = 0$ and $q = [1:0] = \infty$.

$(s_0)_0 =$ divisor of zeros of $s_0 \quad = 2p + q (s_1)_0 =$ divisor of zeros of $s_1 \quad = p + 2q (s_2)_0 =$ divisor of zeros of $s_2 \quad = 3q$

The rational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ given by $[s:t] \mapsto [s^2t : st^2 : t^3]$ is equivalent (i.e. agrees on every open set where both are defined) to the map $[s:t] \mapsto [s^2 : st : t^2]$, which is the Veronese embedding of degree 2. So the morphism is "really given by" $\mathcal{L} = \mathcal{O}(2)$ with sections s^2, st, t^2. The corresponding linear system is the divisor of zeros of $\{as^2+bst+ct^2\}_{[a:b:c]\in\mathbb{P}^2_k} =$ All pairs of points $(p+q)$ including $2p$ on $\mathbb{P}^1$.

Definition 11.47. A **complete linear system** $|D_0|$ is the collection of effective divisiors linearly equivalent to D_0.

$$\text{Divisors of } |D_0| \Leftrightarrow \text{ Divisors of zeros of } s \in \Gamma(X, \mathcal{O}(D_0))$$

Curve Classification: Let X be a smooth projective curve over $k = \overline{k}$

(1) If X is genus 0, then $X \cong \mathbb{P}^1$
(2) If X is genus 1, then $X \xrightarrow{\varphi} \mathbb{P}^1$ is a degree 2 branched cover of $\mathbb{P}^1$ ramified at 4 points and $X \cong \mathbb{V}(F_3) \subseteq \mathbb{P}^2$, where F_3 is a smooth cubic.
(3) If X is genus 2, then by RR

$$h^0(K_X) = degK_X + 1 - g(X) + h^0(K_X - K_X) = degK_X + 1 - 2 + 1 = degK_X = 2g(X) - 2 = 2.$$

Let $s_0, s_1 \in \Gamma(X, \omega_X)$. This defines a map $X \longrightarrow \mathbb{P}^1$ given by $x \mapsto [s_0(x) : s_1(x)]$, which is a degree 2 cover of $\mathbb{P}^1$ ramified at 6 points. We determine the number of ramified points using the Hurewitz formula: 2(genus of X) $-2 = n$(2(genus of $\mathbb{P}^1$) $-$ 2) $+ \sum_{p\in X}(e_p - 1)$

$$2 = 2(-2) + \sum_{p\in X}(e_p - 1) \Rightarrow \sum_{p\in X}(e_p - 1) = 6 \Rightarrow \#\text{ of ramified points is 6}$$

We want to show that a curve of genus 1 can be embedded into $\mathbb{P}^2$. Try the divisor $D = 3p_0$. By RR, $(K_X = 2g(X) - 2 = 0)$:

$$h^0(D) = degD + 1 - g(Z) + h^0(K_X - D) = 3 + 1 - 1 + h^0(-D) = 3$$

Take $s_0, s_1, s_2 \in \Gamma(\mathcal{O}(3p_0))$, $(s_0)_0 = 3p_0$.

$$X \xhookrightarrow{\varphi_{|3p_0|}} \mathbb{P}^2 = \text{Proj } k[x,y,z] \text{ given by } r \mapsto [s_0(r) : s_1(r) : s_2(r)]$$

Assume that $\varphi_{|3p_0|}$ is an embedding (proved shortly). $\varphi^*\mathcal{O}(1) = \mathcal{L} = \mathcal{O}(3p_0)$ where $\varphi^*(x) = s_0, \varphi^*(y) = s_1, \varphi^*(z) = s_2$. $X = \mathbb{V}(F), F$ is a homogeneous equation in x, y, z. Note that (line in $\mathbb{P}^2$) $\cap X \in |3p_0|$. Therefore F is degree 3. Hence every genus 1 curve is isomorphic to a (smooth) cubic in $\mathbb{P}^2$.

Lecture 27. April 16, 2009

Proposition 11.48 (Hartshorne II: 7.3)**.** *Let X be a variety over k. Let $X \xrightarrow{\varphi} \mathbb{P}^n$ be a morphism given by $s_0, \ldots, s_n \in H^0(X, \mathcal{L})$ global generators and span $H^0(X, \mathcal{L})$ over k. Let $\mathcal{L} = \mathcal{O}(D_0)$. Then φ is an embedding (i.e. $\mathcal{L}$ is very ample) if:*

(1) *$\mathcal{L}$ separates points: $\forall p, q \in X, \exists s \in H^0(X, \mathcal{L})$ such that $s(p) = 0$ but $s(q) \neq 0$.*
Equivalently, $\exists D \in |D_0| =$(all divisors effective and inearly equivalent to D) such that $p \in D$ and $q \notin D$.
(2) *$\mathcal{L}$ separates tangent vectors: $\forall p$, the stalks of $s \in H^0(X, \mathcal{L})$ which vanish at p span $m_p/m_p^2 \otimes \mathcal{L}$.*
Equivalently, for all tangent directionss v at p, $\exists D \in |D_0|$ such that v is tangent to D at p.

Remark 11.49. $\mathcal{O}_X(D)$ is globally generated is equivalent to $|D|$ is base point free. $\mathcal{O}_X(D)$ is very ample is equivalent to $|D|$ is very ample.

Proposition 11.50. *Let D be a divisor on a curve X.*

(1) *$\mathcal{O}_X(D)$ is globally generated* $\Leftrightarrow \forall p \in X, dim H^0(X, D-p) = dim H^0(X,D) - 1$.
This is equivalent to: $|D|$ *is base point free* $\Leftrightarrow \forall p \in X, h^0(D-p) = h^0(D) - 1$.
(2) *$\mathcal{O}_X(D)$ is very ample* $\Leftrightarrow \forall p, q \in X$ *(including $p = q$)*, $dim H^0(X, D-p-q) = dim H^0(X,D) - 2$.
This is equivalent to: $|D|$ *is very ample* $\Leftrightarrow \forall p, q \in X, h^0(D-p-q) = h^0(D) - 2$.

Corollary 11.51. *Let D be a divisor on a genus g curve. Then*

$$\mathcal{O}_X(D) \textit{ is globally generated if } degD \geq 2g$$

$$\mathcal{O}_X(D) \textit{ is very ample if } degD \geq 2g+1$$

Of Corollary. By RR, $h^0(D) = degD + 1 - g(X) + h^0(K_X - D)$. The degree of $K_X - D = 2g-2-degD$, which is always negative under assumptions 1 and 2 so under these assumption $h^0(K_X - D) = 0$. So $h^0(D) = degD + 1 - g$. □

Of Proposition. (This is Prop. 3.1 in Hartshorne IV)
(1) $p \overset{i}{\hookrightarrow} X$

$$0 \longrightarrow \mathcal{O}(-p) \longrightarrow \mathcal{O}_X \xrightarrow{\text{eval.}} i_*\mathcal{O}_p \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_X(D-p) \longrightarrow \mathcal{O}_X(D) \xrightarrow{\text{eval.}} \mathcal{O}_p \otimes \mathcal{O}_X(D) = \mathcal{O}_X(D)/m_p\mathcal{O}_X(D) \longrightarrow 0$$

$$0 \longrightarrow H^0(X, D-p) \longrightarrow H^0(X,D) \xrightarrow{\text{eval.}} k \longrightarrow H^1(X, D-p) \longrightarrow \dots$$

$\mathcal{O}_X(D)$ is globally generated at $p \Leftrightarrow e$ is surjective. This shows that either $h^0(D-p) = h^0(D) - 1$ or p is a basepoint.
(2) Using prop. 7.3 in Hartshorne, show ($\Leftarrow$): Need $\mathcal{O}(D)$ separates points and tangent vectors.

$$h^0(D-p-q) = h^0(D) - 2 \Rightarrow h^0(D-p) = h^0(D) - 1$$

By *(1)*, $\mathcal{O}_X(D)$ is globally generated. To separate points: fix $p, q \in X$. Need † $s \in H^0(X, \mathcal{O}_X(D))$ such that $s(p) = 0$ but $s(q) \neq 0$. Equivalently, we need $s \in H^0(X, \mathcal{O}_X(D-p)) = H^0(D-p)$. † holds $\Leftrightarrow Q$ is not a base point of $|D-p|$. Since $h^0(D-p-q) = h^0(D-p) - 1 = h^0(D) - 2$ (by *(1)*), Q is not a base point of $|D-p|$. Therefore $|D|$ separates points (i.e. $\mathcal{O}_X(D)$ separates points).
To separate tangent vectors: $\forall p \in X, s \in H^0(X, \mathcal{O}(D))$ which vanishes at p (i.e. $s \in H^0(X, \mathcal{O}_X(D-p)) = H^0(D-p)$). s_p spans $m_p/m_p^2 \otimes \mathcal{O}_X(D) \Leftrightarrow s_p$ does not vanish to order at least 2 at $p \Leftrightarrow$ the corresponding divisor does not have $2p$ in its expression.
Want to show:

$$0 \longrightarrow \mathcal{O}(-p) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_p \longrightarrow 0 \text{ then tensoring on the right by } \mathcal{O}_X(D-p)$$

$$0 \longrightarrow \mathcal{O}(D-2p) \longrightarrow \mathcal{O}_X(D-p) \longrightarrow k \longrightarrow 0$$

D separates tangent vectors at $p \Leftrightarrow h^0(D-2p) = h^0(D-p) - 1 = h^0(D) - 2$. □

X GENUS 1 CURVE:
Fix a base point $p_0 \in X$. This gives a map:

$$X \longrightarrow Pic^0(X) \subseteq PIc(X) \text{ given by } p \mapsto p - p_0,$$

where $Pic^0(X)$ is a subgroup of $Pic(X)$ of the divisors classes of degree 0.

Proposition 11.52. *For a genus 1 curve, this map is a bijection.*

Proof. *(Injective)* For points $p \neq q$, we need $p - p_0 \nsim q - p_0$.
$p - p_0 \sim q - p_0 \Rightarrow p \sim q \Rightarrow X \cong \mathbb{P}^1$, which is a contradiction hence $p - p_0 \nsim q - p_0$.
(Surjective) Take any divisor D of degree 0 on X. Want $D \sim p - p_0$ for some $p \in X$ (i.e. want to show $\exists p$ such that $D + p_0 \sim p$ or, equivalently, $p \in |D + p_0|$). We want $H^0(X, \mathcal{O}(D+p_0))$ to be non-zero. Since $s \in H^0(X, \mathcal{O}(D+p_0)) - \{0\}$, then the corresponding divisor is an effective divisor $D' \sim D + p_0$ of degree $1 \Rightarrow D' = p$. Use RR to check that $h^0(X, \mathcal{O}(D+p_0)) \geq 1$. □

Since $Pic^0(X)$ is a group and there is a bijection between $Pic^0(X)$ and X, we can give a group structure to X as follows: For $p, q \in X$ define $p \oplus q = r$, where r is the unique point satisfying $(p - p_0) + (q - p_0) \sim r - p_0$. Then p_0 will be the identity in X.

Lecture 28. April 21, 2009

For this lecture, a surface is a smooth projective variety over $k = \overline{k}$ of dimension 2. Every complete dimension 2 smooth variety is projective (i.e. the map $X \longrightarrow \text{Spec } k$ is proper). We need tools to map X to projective spaces so we can answer fundamental questions about X.

(1) Toos for finding $h^0(X, D)$ - to get maps to projective space (Formula: RR)
(2) Criteria for check D to be ample (well-understood) or very ample (research area)

By definition, the Euceler characteristic, $\chi(X, \mathcal{O}_X(D))$, is: $\chi(X, \mathcal{O}_X(D)) = h^0(X, \mathcal{O}(D)) - h^1(X, \mathcal{O}(D)) + h^2(X, \mathcal{O}(D))$.

Theorem 11.53. *(RR for Surfaces) Let X be a surface and D a divisor on X. Then*

$$\chi(X, \mathcal{O}_X(D)) = \chi(X, \mathcal{O}_X) + \frac{1}{2} D \cdot (D - K_X),$$

where $D \cdot (D - K_X)$ is the intersection number of the two divisors D and $D - K_X$.

Often in applications, $-h^1(X, \mathcal{O}(D)) + h^2(X, \mathcal{O}(D)) = 0$ - "Kodaira Vanishing."

Intersection Pairing on a Surface

Let X be a surface and D, C be two divisors on X. The intersection number $D \cdot C \in \mathbb{Z}$.
Idea: If D and C are smooth curves intersecting transversly, then $D \cdot C = \#$ points in $D \cap C$.

- The intersection number only depends on the divisor class of C and D (i.e. $D_1 \sim D_2, D_1 \cdot C = D_2 \cdot C, \forall C \in Div(X)$).
- If C, D are smooth irreducible curves, then

$$C \cdot D = \#(C \cap D) \text{ counting multiplicities } = \sum_{p \in C \cap D} (C \cdot D)_p, \text{ where } (C \cdot D)_p = \dim_k \big(\frac{\mathcal{O}_{X,p}}{(f,g)},$$

where f, g are the equations for C, D, respectively.

Definition 11.54. Curves C and D **intersect transversely** at $p \in X$ if $(C \cdot D)_p = 1$.

Example 11.55. Let $X = \mathbb{P}^2$ and C, D be smooth curves of degree d, c intersecting transversly. Then

$$D \cdot C = \#(D \cap C) = dc \textit{by Bazou's theorem}$$

Example 11.56. Let $X = \mathbb{P}^2$ in $x : y : z$, $L = \mathbb{V}(y), C = \mathbb{V}(yz - x^2)$. In U_z, L has local defining equation $\frac{y}{z} = t$ and C has local defining equation $\frac{y}{z} - \left(\frac{x}{z}\right)^2 = t - s^2$.

$$(C \cdot L)_p = \dim_k \frac{\mathcal{O}_{X,p}}{(t, t - s^2} = \dim_k \frac{k[s,t]_{(s,t)}}{(t, t - s^2)} = \dim_k \frac{k[s]_{(s)}}{(s^2)} = 2,$$

which is what we would expect since the degree of C is 2 and of L is 1, so their product is 2.

Theorem 11.57. *Let X be a surface. There is a unique symmetric bilinear map:*

$$Div(X) \times Div(X) \longrightarrow \mathbb{Z} \text{ given by } (C, D) \mapsto C \cdot D$$

such that

(1) *If C, D are smooth and intersect transversely, then $C \cdot D = \#(C \cap D)$.*
(2) *If $C_1 \sim C_2$, then $C_1 \cdot D = C_2 \cdot D$.*

Sketch of Proof. Use the fact that given H very ample and any $D \in Div(X)$, for $n >> 0, (D + nH)$ is very ample.
For $D_1, D_2 \in Div(X)$, take $D' \in |D_1 = nH|$, which is very ample by the above fact. So $D' \sim D_1 + nH$. Then $D_1 \cdot D_2 = (D' - nH) \cdot D_2$ and we can manipulate the divisors to make them transverse, then apply the first result. □

Fact: If $X \xrightarrow{f} Y$ is a birational morphism of surfaces and $C, D \in Div(Y)$, then $f^*D \cdot f^*C = D \cdot C$.

Example 11.58. Let $X = \mathbb{P}^2$ blownup at a point $P = [0 : 0 : 1]$. Recall that the divisor class group on $\mathbb{P}^2$ is $\mathbb{Z}$ and removing a variety of codimension 2 or higher does not effect this. So the divisor class group of X is $\mathbb{Z} \times \mathbb{Z}$. Let $\tilde{L}_1$ and E be the generators, where E corresponds to the copy of $\mathbb{P}^1$ that was added to $\mathbb{P}^2$ to get X.

$$X \xrightarrow{\pi} \mathbb{P}^2 \text{ and } \pi(E) = p$$

Let $\pi^*(L_1) = \tilde{L}_1$ and $\pi^*(L_2) = \tilde{L}_2 + E \sim L_2$, where L_1 does not go through p but L_2 does and L_1 intersects L_2 once. Then

$$\tilde{L}_1 \cdot E = 0 \text{ and } \tilde{L}_2 \cdot E = 1 \text{ and } L_1 \cdot L_2 = 1$$

$\tilde{L}_2 + E \sim \tilde{L}_1$, so $(\tilde{L}_2 + E) \cdot E = \tilde{L}_1 \cdot E = 0$. Now $\tilde{L}_2 \cdot E = 1$ since L_2 goes through the point p once in $\mathbb{P}^2$. Therfore $\tilde{L}_2 \cdot E = 1$ so $E^1 = -1$.

Definition 11.59. Let X be a smooth surface. Divisors D_1, D_2 are **numerically equivalent**, denoted $D_1 \equiv D_2$, if $D_1 \cdot C = D_2$
$cdotC, \forall C \in Div(X)$.

$D_1 \sim D_2 \Rightarrow D_1 \equiv D_2$, so numerical equivalence is coarser than $\sim$. This induces a symmetric bilinear non-degenerate pairing:

$$Div(X)/\equiv \times Div(X)/\equiv \longrightarrow \mathbb{Z}$$

Now (non-obvious) fact is that $Div(X)/\equiv$ is a finitely-generated free abelian group.
Neron-Severi Space:

$$Div(X)/\equiv \otimes_{\mathbb{Z}} \mathbb{R} \text{ is a vector space over } \mathbb{R} \text{ of finite dimension}$$

The intersection pairing gives a non-degenerate symmetric bilinear form $(\cdot, \cdot)$.

Remark 11.60. After choosing an appropriate basis for the Neron-Severi space $NS(X) = Div(X)/\equiv \otimes_{\mathbb{Z}} \mathbb{R}$, this form has 1 in the first diagonal entry and negative ones in the rest of the diagonal *(Hodge index theorem)*.

Theorem 11.61. *(Nakai-Moishezon Criterion) Let X be a smooth surface and $D \in Div(X)$. D is ample $\Leftrightarrow D \cdot C > 0$, for all effective, irreducible divisors C. $D^2 > 0$.*

Theorem 11.62. *(Kleiman) Let X be a smooth surface and $D \in Div(X)$. D is ample $\Leftrightarrow D \cdot C > 0, \forall C$ in the closed cone of effective divisors on X.*

The cone of effective divisors on $X \subseteq NS(X) = Div(X)/\equiv \otimes_{\mathbb{Z}} \mathbb{R}$ and it is the divisor class represented by effective curves where $|D| \neq 0$.