Complex Function Theory:
Analysis on Domains in $\mathbb{CP}^n$

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Contents

[0]Remark: These notes were typed during lecture and edited somewhat, so be aware that they are not error free. if you notice typos, feel free to email corrections to swlapan@umich.edu.

1. One Complex Variable

Lecture 1. September 9, 2009

Definition 1.1. Let V and W be vector spaces over $\mathbb{C}$. A **pairing** is a bilinear map:

$$V \times W \longrightarrow \mathbb{C}.$$

Let γ be a C^1 simple closed curve in $\mathbb{C}$. Let Ω_+ be the interior of the closed curve γ and Ω_- the exterior.

$$A(\Omega_+) \equiv \{f \text{ cont. on } \Omega_+ \cup \gamma \mid f \text{ holo. on } \Omega_+\}$$
$$A(\Omega_-) \equiv \{f \text{ cont. on } \Omega_- \cup \gamma \cup \{\infty\} \mid f \text{ holo. on } \Omega_-, f(\infty) = 0\}$$

Given $f, g \in A(\Omega_+)$, how can we pair them over γ? In other words, we want a bilinear map $A(\Omega_+) \times A(\Omega_+) \longrightarrow \mathbb{C}$ which, ideally, is symmetric and non-trivial.

1st try: $\int_\gamma fg dz$. This is symmetric, but always zero.

2nd try: $\int_\gamma f\overline{g} dz$. This is usually non-zero (unless $f \equiv 0$ or g constant), but not symmetric.

3rd try: $\int_\gamma f\overline{g}|dz| = \int_\gamma f\overline{g} ds$ (where s is the arc length). This not only satisfies all of the requirements, but also is a good pairing on both $A(\Omega_+)$ and $A(\Omega_-)$. Note that this is the inner product on $L^2(\gamma, ds)$.

Let $H_\pm(\gamma) \equiv L^2(\gamma, ds)$ – closure of $A(\Omega_\pm)$. Note that this is a Hardy space.

Example 1.2. Let γ be the unit circle traversed counter-clockwise ($\gamma(\theta) = e^{i\theta}, 0 \le \theta < 2\pi$).

$$L^2(\gamma, ds) = \left\{ \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \mid \sum |a_n|^2 < \infty \right\}$$

In Ω_+ (the interior of γ) z^n always converges for $n \ge 0$ but not for $n < 0$, so:

$$H_+(\gamma) = \left\{ \sum_{n=0}^{\infty} a_n e^{in\theta} \mid \sum |a_n|^2 < \infty \right\}$$

In Ω_- (the exterior of γ) z^n always converges for $n < 0$ but not for $n \ge 0$ (in particular z^n does not converge for $n = 0$ as $||z|| \to \infty$ since $\left|\int_{\Omega_-} ds\right| = \infty$), so:

$$H_-(\gamma) = \left\{ \sum_{n=-\infty}^{-1} a_n e^{in\theta} \mid \sum |a_n|^2 < \infty \right\}$$

Now suppose $f \in H_+(\gamma)$ and $g \in H_-(\gamma)$. How can we pair them?

1st try: $\int_\gamma f\overline{g} ds$. If γ is the unit circle, then $\int_\gamma f\overline{g} ds = 0$:

$$\begin{aligned}
f\overline{g} &= \left(\sum_{n=0}^{\infty} a_n e^{in\theta}\right)\left(\sum_{n=-\infty}^{-1} \overline{b_n} e^{-in\theta}\right) \\
&= \left(\sum_{n=0}^{\infty} a_n e^{in\theta}\right)\left(\sum_{n=1}^{\infty} \overline{b_{-n}} e^{in\theta}\right) \\
&= \sum_{n=1}^{\infty} c_n e^{in\theta} \\
\int_\gamma f\overline{g} &= \int_\gamma \sum_{n=1}^{\infty} c_n e^{in\theta} \\
&= 2\pi i \cdot 0 \\
&= 0
\end{aligned}$$

2nd try: $\int_\gamma fgdz$. If γ is the unit circle, then

$$f = \sum_{n=0}^{\infty} a_n e^{in\theta} \text{ and } g = \sum_{n=-\infty}^{-1} b_n e^{in\theta}$$

and $\int_\gamma fgdz = 2\pi \sum_{n=0}^{\infty} a_n b_{-1-n}$, so this is a good pairing:

$$fg = \left(\sum_{n=0}^{\infty} a_n e^{in\theta}\right)\left(\sum_{n=-\infty}^{-1} \overline{b_n} e^{in\theta}\right)$$
$$= \sum_{n=-\infty}^{-2} c_n e^{in\theta} + \sum_{k=0}^{\infty} a_k b_{-1-k} e^{-i\theta} + \sum_{n=0}^{\infty} c_n e^{in\theta}$$
$$\int_\gamma fgdz = 2\pi i \sum_{k=0}^{\infty} a_k b_{-1-k} e^{-i\theta}$$

The "perfect duality pairing" is given by the following (note: all of the norms are L^2 norms):

$$\text{Let } ||f|| = sup_{||g|| \leq 1} \left| \int_\gamma fgdz \right| \text{ and } ||g|| = sup_{||f|| \leq 1} \left| \int_\gamma fgdz \right|$$

Remark 1.3. There is a perfect duality pairing if and only if γ is a circle.

Now consider the Cauchy integral. Let f be continuous on γ, then

$$Cf(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z}$$

Facts:

(1) Cf is holomorphic on $\mathbb{C} \cup \{\infty\} \setminus \gamma$
(2) $Cf(\infty) = 0$
(3) If $f \in A(\Omega_+)$, then the Cauchy integral formula says that

$$Cf(z) = \begin{cases} f & \text{on } \Omega_+ \\ 0 & \text{on } \Omega_- \end{cases}$$

(4) If $f \in A(\Omega_-)$, then the Cauchy integral formula says that

$$Cf(z) = \begin{cases} 0 & \text{on } \Omega_+ \\ -f & \text{on } \Omega_- \end{cases}$$

(5) If f is C^1, then:
- Cf extends continuously from Ω_+ to $\Omega_+ \cup \gamma$. Let $C_+ f$ be the boundary value of f (i.e. the value on the extension to γ).
- Cf extends continuously from Ω_- to $\Omega_- \cup \gamma$. Let $C_- f$ be the boundary value of f (i.e. the value on the extension to γ).

(6) $C_\pm$ extends to bounded operators from $L^2(\gamma)$ to $H_\pm(\gamma)$
(7) $||C_\pm||$ is the operator norm of $C_\pm$

$$||C_+|| = sup_{||f||_{L^2} \leq 1} ||C_+ f||_{L^2} = ||C_-|| = \left(inf_{f \in H_-, ||f||=1} sup_{g \in H_-, ||g||=1} | \int fgdz| \right)^{-1}$$

(8) $f = C_+ f - C_- f$ on γ

Lecture 2. September 11, 2009

In $\mathbb{C}^1$*:* Let γ be a C^1 simple closed curve, let Ω_+ be the interior of the closed curve γ, and let Ω_- the exterior. Let

$$H_+(\gamma) = \{f \in L^2(\gamma, ds) \mid f \text{ extends ''holo. in } \Omega_+\text{''}\}$$

$$H_-(\gamma) = \{f \in L^2(\gamma, ds) \mid f \text{ extends ''holo. in } \Omega_-\text{'' and } f(\infty) = 0\}$$

H_+ and H_- are Hardy spaces. For $f \in H_+$ and $g \in H_-$, we found that a good pairing of f and g is given by $\int_\gamma fg ds$.

In $\mathbb{C}^n$*:* Now consider higher dimensions: Let S be a sphere (or similar to a sphere) with Ω_+ as the interior and Ω_- as the exterior. The definition of $H_+(S)$ is clear (it follows from the previous definition), but the definition of $H_-(S)$ is not. If $H_-(S)$ is defined as above, then a holomorphic function on Ω_- extends to an entire function and the condition $f(\infty) = 0$ results in the only possibility being the zero function (i.e. $H_-(S) = \{0\}$). Dropping the condition $f(\infty) = 0$ still results in $H_-(S) \subset H_+(S)$. Nevertheless, facts (1)-(7) from the previous lecture do generalize, but (8) does not. We need to a construction of $H_-(S)$ that is "different in higher dimensions, but the same in dimension 1." Construct dual S^* in dimension 1, $S^* \neq S$. This will be discussed more later.

More about $Cf(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z}$.
Assume that γ is C^1 , f is C^1 on γ, and $z \in \Omega_-$. Extend f to a C^1 on $\Omega_+ \cup \gamma$. Then, using Green's theorem, on Ω_-:

$$Cf(z) = \frac{1}{\pi}\int\int_{\Omega_+} \frac{\frac{\partial f}{\partial \bar{\zeta}}}{\zeta - z} dA(\zeta)$$

By the dominated convergence theorem, this is a continuous function for $z \in \Omega_- \cup \gamma$ (note: this integral is convergent towards γ by dominated convergence, but the previous definition of Cf is divergent towards γ). The boundary value of Cf is $C_- f \in H_-(\gamma)$. Now let z be a point on the boundary (i.e. $z \in \gamma$). Around the point $z \in \gamma$, remove a small semi-circle inside Ω_+ of radius ϵ. Let $\gamma_{+,\epsilon}$ be the curve of the semi-circle around z. Let $\gamma_{1,\epsilon}$ be the curve γ everywhere except the ϵ-neighborhood of z and in that neighborhood it is $\gamma_{+,\epsilon}$. Let the interior of $\gamma_{1,\epsilon}$ be $\Omega_{+,\epsilon}$.

$$\begin{aligned} C_- f(z) &= \lim_{\epsilon \to 0} \frac{1}{\pi}\int\int_{\Omega_{+,\epsilon}} \frac{\frac{\partial f}{\partial \bar{\zeta}}}{\zeta - z} dA \\ &= \lim_{\epsilon \to 0} \frac{1}{2\pi i}\int_{\gamma_{+,\epsilon}} \frac{f(\zeta)d\zeta}{\zeta - z} + \lim_{\epsilon \to 0} \frac{1}{2\pi i}\int_{\gamma_{1,\epsilon}} \frac{f(\zeta)d\zeta}{\zeta - z} \\ &= -\frac{f(z)}{2} + \left(\frac{1}{2\pi i} p.v. \int \frac{f(\zeta)d\zeta}{\zeta - z}\right) \end{aligned}$$

where the part in parenthesis is a definition and p.v. stands for principal value.

Exercise 1.4. Show that $Cf(z)$ extends from Ω_+ to $\Omega_+ \cup \gamma$. Call the boundary value $C_+ f \in H_+(\gamma)$. Also show that $C_+ f(z) = \frac{f(z)}{2} + \frac{1}{2\pi i} p.v. \int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z}$.

Corollary 1.5. $C_+ f - C_- f = f$

Theorem 1.6. *$\exists M > 0$ (depending on γ) such that $||C_+ f||_2 \leq M||f||_2, \forall C^1\ f$. The smallest such M is $||C_+||$.*

Corollary 1.7. *C_+ extends to a bounded linear map $L^2(\gamma, ds) \to H_+(\gamma)$.*

What's the role of the Riemann sphere here?

Consider $\mathbb{C} \cup \{\infty\} \xrightarrow{\varphi} \mathbb{C} \cup \{\infty\}$ given by $z \mapsto \varphi(z) = \frac{c+dz}{a+bz}$. Insist that $ad - bc = 1$ (this

determines $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ up to sign). Let $\tilde{\gamma} = \varphi^{-1}(\gamma) = \frac{-c+a\gamma}{d-b\gamma}$ and $\tilde{f}(z) = \frac{f(\varphi(z))}{a+bz}$.
Everything discussed so far is preserved:

$$\int_{\tilde{\gamma}} \tilde{f}\tilde{g}dz = \int_{\gamma} fgdz$$

$$\int_{\tilde{\gamma}} \tilde{f}\overline{\tilde{g}}ds = \int_{\gamma} f\overline{g}ds$$

$$C_{\tilde{\gamma}}\tilde{f} = \tilde{C_{\gamma}f}$$

$$||C_{\tilde{\gamma},+}|| = ||C_{\gamma,+}||$$

Definition 1.8. Let V be a vector space and $P : V \longrightarrow V$ be a linear map. Then P is a **projection operator** if $P^2 = P$.

Exercise 1.9. If P is a projection operator then:

(1) $I - P$ is also a projection operator

Proof. $(I-P)^2 = I - 2P + P^2 = I - P$ □

(2) ker$P$ =range$(I-P)$

Proof. $\forall x \in \ker(P), (I-P)(x) = x \Rightarrow \ker(P) \subseteq \text{range}(I-P)$. For $y \in \text{range}(P)$, $y = (I-P)(x)$ for some x and $P(y) = P((I-P(x)) = P(x) - P^2(x) = 0 \Rightarrow \ker(P) \supseteq \text{range}(I-P)$. □

(3) range$(P) = \ker(I-P)$

Proof. $(I-P)(P(x)) = P(x) - P^2(x) = 0 \Rightarrow \text{range}(P) \subseteq \ker(I-P)$ and if $z \in \ker(I-P)$, then $(I-P)(z) = 0$ so $P(z) = z \Rightarrow \text{range}(P) \supseteq \ker(I-P)$. □

(4) ker $P\cap$ range P=ker$P\cap$ ker$(I-P)$= $\{0\}$

Proof. The first equality follows from (3) and the second equality follows from the following: If $P(x) = 0$ and $(I-P)(x) = 0$, then $0 = (I-P)(x) = x \Rightarrow x = 0$. □

(5) V =ker $P\oplus$ range P

Proof. This follows from (4) and dimensional analysis. □

(6) Given $V = V_1 \oplus V_2$, there is a unique projection operator $P : V \longrightarrow V$ with kernel V_1 and range V_2.

Proof. Let $P(v_1 + v_2) = v_2$. □

Lecture 3. September 14, 2009

Let γ be a C^1 counterclockwise simple closed curve with Ω_+ as the interior and Ω_- as the exterior. Let $\tilde{\gamma}$ be a C^1 clockwise simple closed curve with $\widetilde{\Omega_-}$ as the interior and $\widetilde{\Omega_+}$ as the exterior. Let φ be a map between the tilde-spaces to the regular spaces (view both spaces as the Riemann sphere). This shows that anything we do for Ω_+ can be done for Ω_-. Ω_+ is always the portion bounded by the positively oriented part of the curve and Ω_- is always the portion bounded by the negatively oriented part of the curve.

$$f \in C^1(\gamma) \Rightarrow C_{\pm}f(z) = \pm\frac{f(z)}{2} + \text{p.v.}\frac{1}{2\pi i}\int_{\gamma}\frac{f(\zeta)d\zeta}{\zeta - z} \in A(\Omega_{\pm})$$

$$\mathcal{H}f(z) = \frac{1}{2\pi i}\text{ p.v. }\int_{\gamma}\frac{f(\zeta)d\zeta}{\zeta - z} = \frac{C_+f + C_-f}{2} \in C(\gamma)$$

$$H_\epsilon f(z) \equiv \frac{1}{2\pi i}\int_{\zeta\in\gamma,|\zeta-z|\geq\epsilon}\frac{f(\zeta)d\zeta}{\zeta-z}\xrightarrow{\epsilon\to 0^+}\mathcal{H}f(z) \text{ uniformly}$$

For $f \in A(\Omega_+), C_+f = f$ and $C_-f = 0$ Similarly, for $f \in A(\Omega_-), -C_\pm f = f$ and $C_+f = 0$. If we extend the domain to $H_\pm$, then for $f \in H_\pm(\gamma), \pm C_\pm f = f$ and $C_\mp f = 0$. Therefore $(C_+)^2 f = C_+ f$ and $(-C_-)^2 f = -C_- f$. Hence C_+ and $-C_-$ are both projection operators with:

$$\text{range}(\pm C_\pm) = H_\pm(\gamma)$$
$$\text{kernel}(\pm C_\pm) = \text{range}(I \mp C_\pm) = \text{range}(\mp C_\mp) = H_\mp(\gamma)$$

Recall that $L^2(\gamma, ds) = H_+ \oplus H_-$.

$$\mathcal{H}^2 = \left(\frac{C_+ + C_-}{2}\right)^2 = \frac{C_+^2 + C_+C_- + C_-C_+ + C_-^2}{4} = \frac{C_+ - C_-}{4} = \frac{I}{4}$$

where $C_+C_- + C_-C_+ = 0$ because for $f \in H_\pm$, $C_\mp f = 0$.

Projection Operator: Let V be a vector space and $V = V_1 \oplus V_2$. Define a projection operator $P(v_1 + v_2) = v_2$, where $v_i \in V_i$. Then the range of P is V_2 and the kernel of P is V_1. Every projection operator has this form.

Special Case: Let V be a Hilbert space (e.g. L^2) and V_2 a closed subspace of V. Then $V = V_2^\perp \oplus V_2$. From this set-up we get an orthogonal projection operator given by $P(w + v) = v$ (i.e. $P = P_{V_2}$). With P as above, $(I - P)g$ is an orthogonal projection operator on $V_2^\perp$.

$$< Pf, g >=< Pf, g - (I - P)g >=< Pf, Pg >=< Pf + (I - P)f, Pg >=< f, Pg >$$

Hence P is self-adjoint.

Exercise 1.10. Given P a projection operator, P is an orthogonal projection operator if and only if (by definition) $\ker(P) = (\text{range}P)^\perp$ if and only if P is self-adjoint.

Special Case: Let $P : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a projection operator. Then $\dim V_1 = \dim V_2 = 1$.

Exercise 1.11. $||P|| = \csc\theta$, where θ is the angle between V_1 and V_2.

Exercise 1.12. In general, if P is a non-zero projection operator, then

$$||P|| = \csc\big(\inf\{\text{angle}(v_1, v_2) \mid v_1 \in \text{Ker}P, v_2 \in \text{Range}P\}$$

The portion inside csc is known as the "1st principal angle" of Jordan.

P is an orthogonal projection operator if and only if $||P|| = 1$.
Let $\Omega \subset \mathbb{C}^n$ be open. The **Bergman space** $B(\Omega) = \{f \in L^2(\Omega) \mid f \text{ holo.}\}$.

Proposition 1.13. *$B(\Omega)$ is a closed subspace of $L^2(\Omega)$.*

Proof. Use the solid Mean Value Theorem. □

Let γ be a C^1 counterclockwise, simple closed curve. The **Szego projections** $S_\pm$ are the orthogonal projection operators $L^2(\gamma, ds) \longrightarrow H_\pm(\gamma)$. Recall that $\pm C_\pm : L^2(\gamma, ds) \longrightarrow H_\pm$ are also projection operators.

Theorem 1.14. *$S_\pm = \pm C_\pm \Leftrightarrow C_\pm$ is self-adjoint $\Leftrightarrow ||C_\pm|| = 1 \Leftrightarrow H_+ \perp H_- \Leftrightarrow \gamma$ is a circle or a line. The last "$\Leftrightarrow$" follows from a Kerzman-Stein result.*

We need to prove that $||C_\pm f||_2 \leq M||f||_2$ for $f \in C^1$. This is true for any γ that is C^1, but to simplify the proof we shall assume that γ is C^2. It suffices to show that $||\mathcal{H}f||_2 \leq \widetilde{M}||f||_2$ for $f \in C^1(\Omega)$. Let $u \in C^1(\Omega_+ \cup \gamma)$, where $u(z)$ is the unit tangent vector for γ at $z \in \gamma$ (so γ is a function on S^1 that extends to a $\mathbb{C}$-valued function in the interior of γ).

Exercise 1.15. For $f \in C^1(\gamma) \Rightarrow \frac{1}{2\pi i}\int_{\gamma-\{|\zeta-z|<\epsilon\}}\frac{\overline{u(z)}f(\zeta)u(\zeta)d\bar{\zeta}}{\bar{z}-\bar{\zeta}}$ converges uniformly as $\epsilon \to 0^+$ to $\mathcal{H}^* f \in C^1(\gamma)$. *Hint: follow work from the previous lecture.*

Lecture 4. September 16, 2009

$$\mathcal{H}_\epsilon f(z) = \frac{1}{2\pi i}\int_{\zeta\in\gamma, |\zeta - z|\geq\epsilon} \frac{f(\zeta)d\zeta}{\zeta - z} \longrightarrow \mathcal{H}f(z) \text{ as } \epsilon\to 0 \text{ uniformly}$$

$$\mathcal{H}^*_\epsilon f(z) = \frac{1}{2\pi i}\int_{\zeta\in\gamma, |\zeta - z|\geq\epsilon} \frac{\overline{u(z)}f(\zeta)u(\zeta)d\overline{\zeta}}{\overline{\zeta} - \overline{z}} \longrightarrow \mathcal{H}^* f(z) \text{ as } \epsilon\to 0 \text{ uniformly}$$

where $f\in C^1(\gamma), z\in\gamma, u\in C^1(\Omega_+\cup\gamma)$ and $u(z)$ is a unit tangent vector for γ at $z\in\gamma$.
*NOTE: $|dz| = u(z)d\overline{z}$ and $u(z)|dz| = dz$ on γ.
Define $\delta(\zeta, z) =$ distance from ζ to z along γ (where $\zeta, z\in\gamma$).

Lemma 1.16 (1)**.**

$$\varphi(z,\zeta) = \begin{cases} \frac{\delta(z,\zeta)}{|z-\zeta|}, & z\neq\zeta \\ 1, & z=\zeta \end{cases} \textit{ is continuous on } \gamma\times\gamma$$

Corollary 1.17. *φ is bounded on $\gamma\times\gamma$.*

Lemma 1.18 (2)**.** *Assume that γ is C^2. Then*

$$\frac{u(\zeta)}{\zeta - z} - \frac{1}{\delta(\zeta,z)} \textit{ is bounded on } (\gamma\times\gamma) - \{\zeta = z\}$$

Recall that $u\in C^1(\Omega_+\cup\gamma)$ and $u(z)$ is the unit tangent vector for γ at $z\in\gamma$.

Proof. Parametrize γ by arc length. Let $f:[0,l]\longrightarrow\gamma$ such that $f'(t) = u(f(t))$ and $f(0)$ is far away from z,ζ on γ. Let $\zeta = f(s), z = f(t)$. Then

$$\frac{u(\zeta)}{\zeta - z} - \frac{1}{\delta(\zeta,z)} = \frac{f'(s)}{\zeta - z} - \frac{1}{s-t} = \frac{f'(s)(s-t) - (f(s)-f(t))}{(s-t)^2}\frac{s-t}{\zeta - z}$$

Note $\frac{s-t}{\zeta - z}$ is bounded by lemma 1, so

$$\left|\frac{f'(s)(s-t) - f(t) + f(s)}{(s-t)^2}\right| \leq \frac{\max|f''|}{2}$$

is bounded by Taylor's theorem. ☐

Lemma 1.19 (3)**.** *Suppose $f,g\in C^1(\gamma)$. Then $\int(\mathcal{H}_\epsilon f)\overline{g}ds = \int f\overline{(\mathcal{H}^*_\epsilon g)}ds$.*

Proof. Use Fubini's theorem and *. ☐

Lemma 1.20 (4)**.** *Suppose $f,g\in C^1(\gamma)$. Then $\int(\mathcal{H}f)\overline{g}ds = \int f\overline{(\mathcal{H}^* g)}ds$.*

Proof. Use uniform convergence $\mathcal{H}_\epsilon f\to\mathcal{H}f, \mathcal{H}^*_\epsilon g\to\mathcal{H}^* g$ and lemma 3. ☐

Now look at

$$\begin{aligned}(\mathcal{H}-\mathcal{H}^*)f &= p.v\left(\frac{1}{2\pi i}\int f(\zeta)\left(\frac{d\zeta}{\zeta - z} - \frac{\overline{u(z)}u(\zeta)d\overline{\zeta}}{\overline{\zeta}-\overline{z}}\right)\right) \\ &= p.v\left(\frac{1}{2\pi i}\int f(\zeta)\left(\frac{u(\zeta)}{\zeta - z} - \frac{\overline{u(z)}}{\overline{\zeta}-\overline{z}}\right)|d\zeta|\right)\end{aligned}$$

Lemma 1.21 (5)**.** $\frac{u(\zeta)}{\zeta - z} - \frac{\overline{u(z)}}{\overline{\zeta}-\overline{z}}$ *is bounded on $\gamma\times\gamma - \{\zeta = z\}$.*

Proof.

$$\frac{u(\zeta)}{\zeta - z} - \frac{\overline{u(z)}}{\overline{\zeta}-\overline{z}} = \left(\frac{u(\zeta)}{\zeta - z} - \frac{1}{\delta(\zeta,z)}\right) + \left(\frac{1}{\delta(\zeta,z)} - \frac{\overline{u(z)}}{\overline{\zeta}-\overline{z}}\right)$$

Each term in parenthesis is bounded by lemma 2. ☐

Therefore:

$$(\mathcal{H}-\mathcal{H}^*)f = \frac{1}{2\pi i}\int \left(\frac{u(z)}{\zeta - z} - \frac{\overline{u(z)}}{\overline{\zeta}-\overline{z}}\right) f(\zeta)|d\zeta|$$

(we do not need the principal value)

Corollary 1.22. *$||(\mathcal{H}-\mathcal{H}^*)f||_2 \leq M'||f||_2$. In fact, $||(\mathcal{H}-\mathcal{H}^*)f||_\infty \leq M''||f||_2$.*

Also,

$$\begin{aligned}
||\mathcal{H}f||_2^2 &= <\mathcal{H}f, \mathcal{H}f> \\
&= <\mathcal{H}^*f + (\mathcal{H}-\mathcal{H}^*)f, \mathcal{H}f> \\
&= <f, \mathcal{H}^2 f> + <(\mathcal{H}-\mathcal{H}^*)f, \mathcal{H}f> \\
&= \frac{1}{4}||f||_2^2 + <(\mathcal{H}-\mathcal{H}^*)f, \mathcal{H}f> \\
&\leq \frac{1}{4}||f||_2^2 + M'||f||_2||\mathcal{H}f||_2 \textit{ by Cauchy-Schwarz} \\
&\leq \frac{1}{4}||f||_2^2 + \frac{(M')^2}{2}||f||_2^2 + \frac{||\mathcal{H}f||_2^2}{2}
\end{aligned}$$

where the final inequality uses:

$$0 \leq \frac{1}{2}(M'||f||_2 - ||\mathcal{H}f||_2)^2 = \frac{1}{2}(M')^2||f||_2^2 - M'||f||_2||\mathcal{H}f||_2 + \frac{1}{2}||\mathcal{H}f||_2^2$$

which implies that

$$M'||f||_2||\mathcal{H}f||_2 \leq \frac{(M')^2}{2}||f||_2^2 + \frac{||\mathcal{H}f||_2^2}{2}$$

Then $||\mathcal{H}f||_2^2 \leq \frac{1}{4}||f||_2^2 + \frac{(M')^2}{2}||f||_2^2 + \frac{||\mathcal{H}f||_2^2}{2}$ implies (by rearranging the terms) that

$$||\mathcal{H}f||_2^2 \leq \left(\frac{1}{2} + (M')^2\right)||f||_2^2 \text{ and thus}$$

$$||C_\pm f||_2 = \left|\left| \pm\frac{f}{2} + \mathcal{H}f\right|\right|_2 \leq \left(\frac{1}{2} + \sqrt{\frac{1}{2} + (M')^2}\right)||f||_2$$

So the operators $C_\pm$ are bounded, as previously claimed.

Lecture 5. September 18, 2009

$S_\pm$ (the Szego projections) are characterized by:

(1) $S_\pm$ projects $L^2(\gamma, ds)$ onto $H_\pm(\gamma)$
(2) $\int_\gamma (S_\pm f)\overline{g}ds = \int_\gamma f\overline{(S_\pm g)}ds$

$C_\pm$ (the Cauchy integrals) satisfy condition (1) but not (2).

Proposition 1.23. *$C_\pm$ satisfy:*

(1) *$C_\pm$ projects $L^2(\gamma, ds)$ onto $H_\pm(\gamma)$*
(2) *$\int_\gamma (C_\pm f)g dz = -\int_\gamma f(C_\pm g)dz$*

Proof. Claim (1) is clear so we shall prove claim (2). $f, g \in A(\Omega_+) \Rightarrow \int_\gamma fg dz = 0$ and $f, g \in A(\Omega_-) \Rightarrow \int_\gamma fg dz = 0$ (both f and g have zeros at ∞, so ∞ is a double zero, hence by the exterior residue theorem the residue is zero). Pass to the limit

$$f, g \in H_+(\gamma) \Rightarrow \int_\gamma fg dz = 0 \text{ and } f, g \in H_-(\gamma) \Rightarrow \int_\gamma fg dz = 0$$

$$\int_\gamma (C_\pm f)g dz = \int_\gamma (C_\pm f)(C_+ g - C_- g)dz = \int_\gamma (C_\pm f)(\pm C_\pm g)dz$$

where the second equality follows since either $C_+g = 0$ or $C_-g = 0$. Furthermore,

$$\int_\gamma (C_\pm f)(\pm C_\pm g)dz = \pm \int_\gamma (C_\pm f - C_\mp f)C_\pm g dz = -\int_\gamma f(C_\pm g)dz$$

Therefore,

$$\int_\gamma (C_\pm f)g dz = -\int_\gamma f(C_\pm g)dz$$

□

Exercise 1.24. The properties in the above proposition characterize $C_\pm$.

Let $\tilde{g} = \overline{gu}$, u be a unit vector.

$$\begin{aligned}
&\text{If } f \in L^2(\gamma, ds) \Rightarrow \\
||f|| &= \sup_{g\in L^2, ||g||=1} \left| \int_\gamma f\bar{g} ds \right| \\
&= \sup_{g\in L^2, ||g||=1} \left| \int_\gamma f\overline{gu} dz \right| \\
&= \sup_{\tilde{g}\in L^2, ||\tilde{g}||=1} \left| \int_\gamma f\tilde{g} dz \right| \\
||C_+|| &= \sup_{f\in L^2, ||f||=1} ||C_+ f|| \\
&= \sup_{f\in L^2, ||f||=1, g\in L^2, ||g||=1} \left| \int_\gamma (C_+ f) g dz \right| \\
&= \sup_{f\in L^2, ||f||=1, g\in L^2, ||g||=1} \left| \int_\gamma f(C_- g) dz \right| \\
&= \sup_{g\in L^2, ||g||=1} ||C_- g|| \\
&= ||C_-|| \\
&\text{If } f \in H_+(\gamma) \Rightarrow \\
||f|| &= \sup_{g\in L^2, ||g||=1} \left| \int_\gamma (C_+ f) g dz \right| \\
&= \sup_{g\in L^2, ||g||=1} \left| \int_\gamma f(C_- g) dz \right| \\
&\leq \sup_{h\in H_-, ||h||\leq ||C_-||} \left| \int_\gamma f h dz \right| \\
&= ||C_-|| \sup_{h\in H_-, ||h||=1} \left| \int_\gamma f h dz \right| \\
\frac{||f||}{||C_-||} &\leq \sup_{h\in H_-, ||h||=1} \left| \int_\gamma f h dz \right| \leq ||f||
\end{aligned}$$

This is a non-exact duality pairing.

$$\frac{1}{||C_+||} = \frac{1}{||C_-||} \leq \inf_{f\in H_+, ||f||=1} \sup_{h\in H_-, ||h||=1} \left| \int_\gamma f h dz \right|$$

Theorem 1.25.
$$\frac{1}{||C_-||} = \inf_{f\in H_+, ||f||=1} \sup_{h\in H_-, ||h||=1} \left| \int_\gamma fhdz \right|$$

Proof. $\forall \epsilon > 0$, pick $g \in L^2$ such that $||C_+g|| = 1$ and $||g|| \leq \frac{1}{||C_+||-\epsilon}$. Choose $f = C_+g$ so that:

$$\begin{aligned} \inf_{f\in H_+, ||f||=1} \left(\sup_{h\in H_-, ||h||=1} \left| \int_\gamma fhdz \right| \right) &\leq \sup_{h\in H_-, ||h||=1} \left| \int_\gamma (C_+g)hdz \right| \\ &= \sup_{h\in H_-, ||h||=1} \left| \int_\gamma ghdz \right| \\ &\leq ||g|| \\ &\leq \frac{1}{||C_+|| - \epsilon} \end{aligned}$$

Hence, we have equality. □

2. Projective Space

Let $k = \mathbb{R}$ or $\mathbb{C}$ and let V be a k-vector space. $\mathbb{P}V$ is the set of all k-lines through 0 in V. $k\mathbb{P}^n = \mathbb{P}k^{n+1}$

$$k^{n+1} \setminus \{0\} \longrightarrow k\mathbb{P}^n \text{ given by } a = (a_0, \dots, a_n) \mapsto l_a = [a_0 : \cdots : a_n]$$

$$l_a = L_b \Leftrightarrow b = \lambda a (\lambda \neq 0)$$

Identify $k\mathbb{P}^n$ with $k^{n+1} \setminus \{0\}/(a \sim \lambda a), \forall \lambda \in k \setminus \{0\}$. Subsets of $k\mathbb{P}^n$ can be identified with subsets of $k^{n+1} \setminus \{0\}$ invariant under (non-zero) multiplication. Use the quotient topology.

$$\mathbb{RP}^n \text{ can be identified with the unit sphere in } \mathbb{R}^{n+1}/a \sim -a$$

$$\mathbb{CP}^n \text{ can be identified with the unit sphere in } \mathbb{C}^{n+1}/a \sim e^{i\theta}a$$

From these examples, it is not surprising that $k\mathbb{P}^n$ is compact. In fact, $k\mathbb{P}^n$ is a manifold. Standard charts are called affinizations. Let α be a hyperplane in k^{n+1} with $0 \notin \alpha$ and α_0 be the parallel hyperplane to α through 0. Define $\varphi_\alpha : l \mapsto l \cap \alpha \in \alpha$. Let

$$U_\alpha = \{l \mid l \text{ line through 0 not parallel to } \alpha\} = \{l \mid l \notin \alpha_0\}$$

$U_\alpha \xrightarrow{\varphi_\alpha} \alpha$ is bijective since each line in U_α intersects α once. Note that $\mathbb{P}\alpha_0 = k\mathbb{P}^n \setminus U_\alpha$ and $k\mathbb{P}^n = U_\alpha \sqcup \mathbb{P}\alpha_0$. U_α corresponds to k^n and $\mathbb{P}\alpha$ corresponds to $k\mathbb{P}^{n-1}$. In particular, $\mathbb{CP}^1$ can be identified with $\mathbb{C} \sqcup \{\text{point}\}$ ($\mathbb{R}$-sphere).

Lecture 6. September 21, 2009

$$\varphi(z, \zeta) = \begin{cases} \frac{\delta(z,\zeta)}{|z-\zeta|}, & z \neq \zeta \\ 1, & z = \zeta \end{cases} \text{ is continuous on } \gamma \times \gamma$$

How nice does γ need to be in order for $L^2(\gamma) = H_+(\gamma) \oplus H_-(\gamma)$?

- *Sufficient:* γC^2 (proved in class)
- *Sufficient:* γC^1
- *Sufficient:* γ is Lipschitz
- *Sufficient:* Lemma 1
- *Necessary:* $\frac{\delta(z,\zeta)}{|z-\zeta|}$ is bounded
- *Necessary and Sufficient:* $\exists c > 0$ such that $\text{length}(\gamma) \cap D \leq C(\text{radius})D, \forall \text{disks } D$

Charts:
Let α be a k-hyperplane with $0 \notin \alpha$, α_0 the parallel hyperplane through 0, and

$$U_\alpha = \{l \in k\mathbb{P}^n \mid l \nsubseteq \alpha_0\} \xrightarrow{\varphi_\alpha} \alpha \text{ given by } l \mapsto l \cap \alpha$$

Suppose we have two charts given by α and β. $\varphi_\beta \circ \varphi_\alpha^{-1} : \alpha \setminus \beta_0 \longrightarrow \beta \setminus \alpha_0$

$$k\mathbb{P}^n \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_m} \Leftrightarrow \alpha_{1,0} \cap \cdots \cap \alpha_{m,0} = \{0\}, \text{ where } m \geq n+1$$

The standard atlas is given by:

$$\alpha_j = \{z \in k^{n+1} \mid z_j = 1\}$$

$$\alpha_{j,0} = \{z \in k^{n+1} \mid z_j = 0\}$$

$$\varphi_j : (z_0 : \cdots : z_n) \mapsto \left(\frac{z_0}{z_j}, \ldots, \frac{z_n}{z_j}\right)$$

Replace φ_j by $\widetilde{\varphi}_j : U_{\alpha_j} \longrightarrow \P\alpha_j$ given by $(z_0 : \cdots : z_n) \mapsto (\frac{z_0}{z_j}, \ldots, \hat{1}, \ldots, \frac{z_n}{z_j})$, where the hat means to omit the jth entry.

Definition 2.1. Let $V \subset k^{n+1}$ be an $m+1$-dimensional vector subspace. Then $\mathbb{P}(V)$ is a **projective m-dimensional k-plane**.

Fix affinziation φ_α. There are two possibilities:

(1) $V \subset \alpha_0$ and $\mathbb{P}V \subset \mathbb{P}\alpha_0$ (i.e. "$\mathbb{P}V$ lies at ∞")
(2) $V \nsubseteq \alpha_0, \varphi_\alpha(V) = \alpha \cap V$ affine m-dimensional k-plane in α. $\mathbb{P}V = (\alpha \cap V) \sqcup \mathbb{P}(\alpha_0 \cap V)$. Note that $\mathbb{P}(\alpha_0 \cap V)$ is a projective $m-1$-dimensional k-plane "at ∞"

Special Cases:

(1) $m = n-1$, $\mathbb{P}\alpha_0$ is a projective hyperplane at ∞ with respect to φ_α
(2) $m = 1$, $\mathbb{P}\alpha$ is a hyperplane at ∞ or $\mathbb{P}\alpha =$(affine line)$\sqcup$(one point at ∞)

Exercise 2.2. Two affine lines l_1, l_2 in k^n are parallel if and only if they meet ∞ at the same point.

Definition 2.3. Given an invertible linear map $M : k^{n+1} \longrightarrow k^{n+1}$, there is an induced map $k\mathbb{P}^n \longrightarrow k\mathbb{P}^n$ given by $l_a \mapsto l_{Ma}$. This induced map is a **projective transformation** (also known as a **projective map**, a **linear fractional transformation**, or a **Mobius transformation**).

How do these look in affine coordinates? Let's look in affine patches where $z_0 \neq 0$.
Let $n = 2$ and let A through I be the entries of the matrix/linear map M. Then:

$$\begin{aligned}(z_1, z_2) &\mapsto (1 : z_1 : z_2)\\ &\mapsto (A + Bz_1 + Cz_2 : D + Ez_1 + Fz_2 : G + Hz_1 + Iz_2)\\ &\mapsto \left(\frac{D + Ez_1 + Fz_2}{A + Bz_1 + Cz_2}, \frac{G + Hz_1 + Iz_2}{A + Bz_1 + Cz_2}\right)\end{aligned}$$

Example 2.4. $(z_1, z_2) \mapsto (\frac{1}{z_1}, \frac{z_2}{z_1})$ is a LFT corresponding to $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Example 2.5. $(z_1, z_2) \mapsto (\frac{1}{z_1}, z_2)$ is not a LFT (it is a birational map). It corresponds to the map: $(z_0 : z_1 : z_2) \mapsto (z_1 : z_0 : z_1 z_2)$, which is not defined at $(0 : 0 : 1)$.

Exercise 2.6. The maps $\{\varphi_\beta \circ \varphi_\alpha^{-1}\}_{\alpha,\beta}$ are linear fractional transformations.

M_1, M_2 induce the same linear fractional transformation if and only if $M_1 = \lambda M_2 (\lambda \neq 0)$.
So can restrict to $M \in SL(n+1, k)$.
If $k = \mathbb{R}, n$ even, then get a unique M for each LFT
If $k = \mathbb{R}, n$ odd, then LFT determines M up to sign
If $k = \mathbb{C}$, then LFT determines M up to $(n+1)$st roots of unity.

Example 2.7. The LFT $(z_1, z_2) \mapsto (\frac{1}{z_1}, \frac{z_2}{z_1}$ corresponds to $\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Proposition 2.8.
(1) *LFTs map projective m-dimensional planes to projective m-dimensional planes.*
(2) *Any projective m-dimensional plane can be mapped to any other m-dimensional plane by a LFT.*
(3) *Any projective hyperplane can be mapped to a projective hyperplane at infinity by a LFT.*

Lecture 7. September 23, 2009

Focus on $\mathbb{RP}^n$:
$\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$

$$\begin{aligned} E \subset \mathbb{R} \text{ is } \textbf{convex} &\Leftrightarrow E \text{ is connected} \\ &\Leftrightarrow \mathbb{RP}^1 \setminus E \text{ is connected} \\ &\Rightarrow \mathbb{R} \setminus E \text{ has 0,1, or 2 components} \\ E \subset \mathbb{RP}^1 \text{ is } \textbf{projectively convex} &\Leftrightarrow E \text{ is connected} \\ &\Leftrightarrow \mathbb{RP}^1 \setminus E \text{ is projectively convex} \end{aligned}$$

Let $n > 1, \mathbb{RP}^n = \mathbb{R}^n \cup$ "projective hyperplane at ∞".

Definition 2.9. $E \subset \mathbb{R}^n$ is **convex** if $E \cap l$ is connected for all affine lines $l \subset \mathbb{R}^n$.

Definition 2.10. $E \subset \mathbb{RP}^n$ is **projectively convex** if $E \cap l$ is connected for all projective lines $l \subset \mathbb{RP}^n$.

So $E \subset \mathbb{RP}^n$ is projectively convex $\Leftrightarrow \mathbb{RP}^n \setminus E$ is projectively convex.

Example 2.11. The open/closed ball B in $\mathbb{R}^n$ is convex, but $\mathbb{R}^n \setminus B$ is neither convex nor projectively convex. However, $\mathbb{RP}^n \setminus B$ is projectively convex *(not APS-convex)*.

Definition 2.12 (APS)**.** $E \subset \mathbb{RP}^n$ is **convex** if E is projectively convex and E contains no projective line.

Note: if $E \subset \mathbb{R}^n$, this definition is compatible with the standard definition because for E to contain a projective line it must contain a point at infinity.

Proposition 2.13. *If $E \subset \mathbb{RP}^n$ is convex and ψ is a LFT, then $\psi(E)$ is convex.*

We will show that if E is open/closed in $\mathbb{RP}^n$, then E is convex $\Leftrightarrow \exists \psi$, a LFT, such that $\psi(E) \subset \mathbb{RP}^n$ is convex.

Proposition 2.14. *Suppose that $\alpha \subset \mathbb{RP}^n$ is a projective hyperplane and $\{E_j\}$ is a family of convex subsets of $\mathbb{RP}^n \setminus \alpha$. Then $\cap E_j$ is convex.*

Proof. Move α to infinity so that $\mathbb{RP}^n \setminus \alpha \cong \mathbb{R}^n$. Then quote a standard fact for $\mathbb{R}^n$. □

Consider l_1, l_2 distinct projective lines in $\mathbb{RP}^2$.

Case 1:: l_2 is a line at ∞. Then $\mathbb{RP}^2 \setminus (l_1 \cup l_2) = \mathbb{R}^2 \setminus l_2$ has two components.

Case 2:: Neither line is at ∞. Then $\mathbb{R}^2\backslash(l_1 \cup l_2)$ has four components and $\mathbb{RP}^2\backslash(l_1 \cup l_2)$ has two components.

The same kind of reasoning works in higher dimensions.
If $\alpha_1, \alpha_2 \subset \mathbb{RP}^n$ are distinct projective hyperplanes, then $\mathbb{RP}^n\backslash(\alpha_1 \cup \alpha_2)$ has two components called **open half-spaces**. (Another way to see this is by moving one of the α_i to ∞.) A **closed half-space** is the open half-space union $(\alpha_1 \cup \alpha_2$. Open half-spaces are convex. Closed half-spaces are projectively convex but not convex. A half-space is **affine** if α_1 or α_2 lies at ∞.

Theorem 2.15 (Affine Finite-Dimensional Hahn-Banach Theorem)**.** *Suppose $E \subset \mathbb{R}^n$ is an open convex subset. Then $\mathbb{R}^n \setminus E$ is a (probably infinite or uncountable) union of affine hyperplane. Equivalent to the condition:*

$$(*)\ X_0 \notin E \Rightarrow X_0 \textit{ is in an affine hyperplane disjoint from } E$$

Corollary 2.16. *Let E be a convex closed subset of $\mathbb{R}^n$. Then $\mathbb{R}^n \setminus E$ is a union of affine hyperplanes.*

Proof. For $\epsilon > 0$, let $E_\epsilon = \{x \in \mathbb{R}^n \mid \mathrm{dist}(x, E) < \epsilon\}$. Exercise: E_ϵ is open and convex. Then $E = \cap_{\epsilon>0} E_\epsilon$ and $\mathbb{R}^n \setminus E = \cup_{\epsilon>0} \mathbb{R}^n \setminus E_\epsilon = \mathbb{R}^n \setminus \cap_{\epsilon>0} E_\epsilon$ is a union of affine hyperplanes. □

Remark 2.17. This can fail for E convex but neither open nor closed.

Corollary 2.18. *If $E \subset \mathbb{R}^n$ is a convex set that is either open or closed, then E is the intersection of open affine half-spaces.*

Proof. Pick $x_0 \notin E$. Then x_0 is contained in an affine hyperplane, α_{x_0}, disjoint from E. If E is connected, then E lies on one side of α_{x_0}. Let H_{x_0} be the open half-space bounded by α_{x_0}, i.e. $x_0 \notin H_{x_0} \supset E$. Then $E = \cap_{x_0 \in \mathbb{R}^n \setminus E} H_{x_0} = \cap_{x_0 \in \mathbb{R}^n \setminus E} \mathbb{R}^n \setminus \alpha_{x_0} = \mathbb{R}^n \setminus \cup_{x_0 \in \mathbb{R}^n \setminus E} \alpha_{x_0}$. □

Exercise 2.19. Suppose $E \subset \mathbb{R}^n$ is a convex, closed set. Then E is the intersection of closed affine half-spaces. This is not true if E is open.

All of these results are finite-dimensional versions of the Hahn-Banach theorem.

Proof Of Theorem (*). After translation, we can assume that $x_0 = 0$. If $n = 1$ the proof is easy. Proof by induction:
If $n = 2$: Let $S^1 \subset \mathbb{R}^2$ be a unit circle. Assume that $E \subsetneq \mathbb{R}^n$ and (wlog) $0 \notin E$. (If $0 \in E$, then move the origin to a point in $\mathbb{R}^n \setminus E$). Let $F = \{x \in S^1 \mid \text{ray from } 0 \text{ to } x \text{ hits } E\}$. F is open and connected, so $-F$ is open and connected. Since $0 \notin E$, $x \in F \Rightarrow x \notin -F$ and vice versa, so $F \cap (-F) = \varnothing$ and $F \cup (-F) \subsetneq S^1$. Pick $x \in S^1 \setminus (F \cup (-F))$ and let l_x be the line through 0 and x. Then, as desired, l_x is disjoint from E.
If $n > 2$: $0 \in V \subset \mathbb{R}^n$ any 2-dimensional subspace. Pick a 1-dimensional subset $V_1 \subset V$ containing 0 and satisfying $V_1 \cap E = \emptyset$. Let $\rho : \mathbb{R}^n \longrightarrow \mathbb{R}^n \setminus V_1$ be a projection map. $\rho(E)$ is an open, connected, convex set not containing 0. Pick $0 \in \alpha \subset \mathbb{R}^n \setminus V$, where α is a hyperplane and $\alpha \cap \rho(E) = \emptyset$. Then $0 \in \rho^{-1}(\alpha) \subset \mathbb{R}^n$, note $\rho^{-1}(\alpha)$ is a hyperplane, $\rho^{-1}(\alpha) \cap E = \emptyset$. □

Lecture 8. September 25, 2009

$$\begin{aligned} E \text{ is projectively convex } &\Leftrightarrow E \cap l \text{ is connected }, \forall \text{ projective lines } l \\ &\Leftrightarrow \forall p \neq q \in E, E \text{ contains at least one line segment joining } p \text{ and } q \\ &\Leftrightarrow \mathbb{RP}^n \setminus E \text{ is projectively convex} \end{aligned}$$

Proposition 2.20 (1)**.** *Let $E \subset \mathbb{RP}^n$ be a projectively convex set not contained in a projective hyperplane. Then $E \subset \overline{Int(E)}$ (i.e. E is "fat").*

For a projective hyperplane E, there is a basis $a_1, \dots, a_{n+1}$ for $\mathbb{R}^{n+1}$ such that $l_{a_j} \in E$.

Proof. For $n = 1$, the proof is easy. Use induction.
If $n > 1$: For $p \in E$, we must show that $p \in \overline{\text{Int}(E)}$. Choose a basis $a_1, \dots, a_{n+1}$ of $\mathbb{R}^{n+1}$ with $l_{a_j} \in E, p \in l_{a_1}$. Let $A = \mathbb{P}(\text{span}(a_1, \dots, a_n))$. So A is a hyperplane through p and $A \cap E$ is not contained in a lower-dimensional projective plane. By induction, $p \in \overline{\text{Int}_A(A \cap E)}$. For any point $q \in \text{Int}_A(A \cap E)$, we must show that $q \in \overline{\text{Int}(E)}$. Pick a projective hyperplane B distinct from A so that $B \cap E$ is not contained in a lower-dimensional projective plane. Let $B = \mathbb{P}(\text{span}(a_2, \dots, a_{n+1}))$. Pick $r \in \text{Int}_B(B \cap E) \setminus A$, which is non-empty by the induction hypothesis. Let l be the line through q and r.
Case 1: l and all neighboring lines are in E. Then $q \in \text{Int}(E)$.
Case 2: Perturb the points q, r so that $l \nsubseteq E, r \notin B, q \notin A$. Since $q, r \in E$ and E is projectively convex, there is a line segment joining these points which is contained in E. By perturbing more, we end up getting that $q \in \overline{\text{Int}(E)}$. □

Proposition 2.21 (2)**.** *If $E \subset \mathbb{RP}^n$ is projectively convex, then so are Int(E) and $\overline{E}$.*

Proof. Suppose $\text{Int}(E) \cap l$ is not connected. Imagine a circle, l, with $p_1, p_2 \in \text{Int}(E)$ at $-\frac{\pi}{2}, \frac{\pi}{2}$ and $q_1, q_2 \notin \text{Int}(E)$ at $0, \pi$. Perturb the points q_1, q_2 to $\widetilde{q_1}, \widetilde{q_2} \notin E$ and p_1, p_2 to $\widetilde{p_1}, \widetilde{p_2} \in \text{Int}(E)$. Then $E \cap l$ is not connected, which contradicts that E is projectively convex. □

$E^c = \mathbb{RP}^n \setminus E$ and $\overline{E} = (\text{Int}(E^c))^c$ is projectively convex since E is projectively convex.

Proposition 2.22 (3)**.** *Let E be projectively convex and l be a projective line meeting Int$E$ and IntE^c. Then $\#(bE \cap l) = 2$.*

Proof. Again view the line l as a circle. Let $p_1 \in \text{Int}E$, $q_3, q_2 \in bE$, $p_2 \in \text{Int}E^c$, and $a_1 \in bE$ be points on l in order (going counterclockwise around the circle). Perturb the line l to $\tilde{l}$ and the points so that q_2, q_3 move to E and E^c, respectively, and the rest of the points stay within their respective sets. Then $E \cap \tilde{l}$ is not connected, which is a contradiction. □

Corollary 2.23 (1)**.** *Suppose that E is projectively convex and $n = \dim E > 1$. Then $\overline{E}$ or $\overline{E^c}$ must contain a projective line.*

Proof. Need l disjoint from either Int$E$ or IntE^c. If Int$E$ or IntE^c is empty, then this easily follows. Suppose the boundary of E has finitely many points (i.e. $\#bE < \infty$), then $\#bE^c = \infty$ and the proof is again easy. So assume that $\#bE, \#bE^c = \infty$. Choose points $p_j \in bE_j$ such that $p_j \to p \in bE$. If the line joining p_j to p, denoted $l_{p_j,p}$, does not meet Int$E$ and IntE^c, then we are done so assume not. Then, by proposition 3, the long segment joining p_j to p lies in either Int$E$ or IntE^c. By taking $j \to \infty$ and passing to an appropriate subsequence, these line segments will converge to a line. Since the line segments are contained in the interior of either E or E^c, passing to a subsequence and taking the limit will produce a line in either $\overline{E}$ or $\overline{E^c}$. □

Lecture 9. September 28, 2009

Corollary 2.24 (2)**.** *If $E \subset \mathbb{RP}^2$ is closed and projectively convex, then E or E^c contains a projective line.*

Proof. Suppose that E contains no projective line.
Case 1: Suppose E is contained in a projective line. After a change of coordinates,
*Case 2;*Prop 1 $\Rightarrow E \subset \overline{\text{Int}E}$. Prop 2 $\Rightarrow$ Int$E$ projecitvely convex. Cor 1 $\Rightarrow \overline{E^c} = (\text{Int}E)^c$ contains a projective line. After a LFT, Int$E$ is an open convex subset of $\mathbb{R}^2$. Affine hyperplane $\Rightarrow$ Int$E$ is intersection of open half-spaces.
*Case 2a:*The half-spaces are all parallel. Then Int$E$ is a half-space or a strip. E does contain

a projective line.
*Case 2b:*Then E^c contains a projective line. ☐

Corollary 2.25 (3)**.** *If $E \subset \mathbb{RP}^2$ is open, projectively convex, then E^c is closed, projectively convex and E or E^c contains a projective line.*

Corollary 2.26 (4)**.** *If $E \subset \mathbb{RP}^2$ is (APS-)convex, open/closed, then E^c is a union of projective lines.*

Proof. By corollaries 2/3, E^c contains a projective line. After LFT, $E \subset \mathbb{R}^2$. Quote affine Hanh-Banach. ☐

Theorem 2.27. *("Projective Hahn-Banach," version 1) E (APS-)convex and open/closed, then E^c is union of projective hyperplanes.*

Let $\mathbb{R}^{n+1} = V_1 \oplus V_2$. Then $\exists! Q : \mathbb{R}^{n+1} \twoheadrightarrow V_2$ projective operator with kernel V_1. This induces $\widetilde{Q} : \mathbb{RP}^n \setminus \mathbb{P}V_a \twoheadrightarrow \mathbb{P}V_2$ given by $l_a \mapsto l_{Qa}$ (note that $\widetilde{Q} = \widetilde{Q}^2$.

Lemma 2.28 (1)**.** *Let $l \subset \mathbb{RP}^n$ be a projecitve line. Then*

$$\widetilde{Q}(l) = \begin{cases} \emptyset & \text{if } l \subset \mathbb{P}V_1 \\ \text{proj. line} & \text{if } l \cap \mathbb{P}V_1 = \emptyset \\ \text{point} & \text{else} \end{cases}$$

Proof. Let $l = \mathbb{P}W, W \subset \mathbb{R}^{n+1}$ (so dimension of W is 2). $\dim W \cap V_1 = 2, 0, 1$. ☐

Similar statement for line segments.

Corollary 2.29. *Let E be projectively convex and open/closed. Then $\widetilde{Q}(E \setminus \mathbb{P}V_1)$ is projectively convex.*

Lemma 2.30 (2)**.** *Let $\dim V_1 = 1$ and $E \subset \mathbb{RP}^n \setminus \mathbb{P}V_1$ be (APS-)convex. Then $\widetilde{Q}(E)$ is convex.*

Proof. Must show that $\widetilde{Q}(E) \not\supseteq \mathbb{P}W, W \subset V_2$ of dimension 2.
$\widetilde{Q}^{-1}(\mathbb{P}W) = \mathbb{P}(V_1 \oplus W) \setminus \mathbb{P}V_1$ (note: $\mathbb{P}(V_1 \oplus W) \cong \mathbb{RP}^2$ and $\mathbb{P}V_1$ is a point in $\mathbb{RP}^2$). By corollary 4, $\exists$ a projective line l such that $\mathbb{P}V_1 \subset l \subset \mathbb{P}(V_1 \oplus W) \setminus E$.
$\widetilde{Q}(l)$ is a point in $\mathbb{P}W \setminus \widetilde{Q}(E)$, so $\mathbb{P}W \not\subseteq \widetilde{Q}(E)$. ☐

Remark 2.31. This is true for higher dimensional V's (prove by induction).

Proof of Hahh-Banach. By induction on n, let $E \subset \mathbb{RP}^n$ be a convex, open/closed subset. Pick $a \notin E$. Want a projective hyperplane containing a in E^c. Pick a projective hyperplane H not containing a. $\widetilde{Q}_{a,H}(E)$ is a convex subset of E. Inductive hypothesis implies that $\exists \widetilde{H} \subset H$ an $(n-2)$-dimensional plane and $\widetilde{H} \cap \widetilde{Q}_{a,H}(E) = \emptyset$. E is disjoint from $\widetilde{Q}^{-1}_{a,H}(\widetilde{H}) \cup \{a\}$ projective hyperplane in $\mathbb{RP}^n$. ☐

Dual Projective Space

$$\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\}/a \sim \lambda a \text{ where elements are column vectors, i.e. } \begin{pmatrix} a_0 \\ \cdot\cdot \\ \vdots \\ \cdot\cdot \\ a_n \end{pmatrix}$$

$$\mathbb{RP}^{n*} = \mathbb{R}^{n+1} \setminus \{0\}/a \sim \lambda a \text{ where elements are row vectors, i.e. } (b_0 : \cdots : b_n)$$

Then $\sum a_j b_j$ is not defined but $\sum a_j b_j = 0$ is a well-defined condition.
$b \in \mathbb{RP}^{n*} \Rightarrow h_b = \{a \in \mathbb{RP}^n \mid \sum a_j b_j = 0\}$ is a projective hyperplane in $\mathbb{RP}^n$.

$\mathbb{RP}^{n*} \leftrightarrow \{\text{proj. hyperplane in } \mathbb{RP}^n\}$
$a \in \mathbb{RP}^n \Rightarrow h_a^* = \{b \in \mathbb{RP}^{n*} \mid \sum a_j b_j = 0\} = \{b \in \mathbb{RP}^{n*} \mid a \in h_b\}$ is a projective hyperplane.
$\mathbb{RP}^n \leftrightarrow \{\text{proj. hyperplane in } \mathbb{RP}^{n*}\}$

Definition 2.32. Let $E \subset \mathbb{RP}^n$. The **polar** of E, denoted E^o, is $\{b \in \mathbb{RP}^{n*} \mid h_b \in E^c\}$.

Facts:

(1) Always have $E \subset E^{oo}$
(2) Theorem above if and only if E convex, open/closed, then $E = E^{oo}$.

Lecture 10. September 30, 2009

$$\mathbb{RP}^n \text{ where elements are column vectors } a = \begin{pmatrix} a_0 \\ \cdot\cdot \\ \vdots \\ \cdot\cdot \\ a_n \end{pmatrix}$$

$$\mathbb{RP}^{n*} = \text{ where elements are row vectors } b = (b_0 : \cdots : b_n)$$

$$ba = \sum_{j=0}^{n} a_j b_j = 0 \Leftrightarrow a \in \mathfrak{h}_b \Leftrightarrow b \in \mathfrak{h}_a^*$$

where $\mathfrak{h}_b$ is a hyperplane in $\mathbb{RP}^n$ and $\mathfrak{h}_a^*$ is the set of hyperplanes through a (i.e. a hyperplane in $\mathbb{RP}^{n*}$). $M = \mathrm{SL}(n+1, \mathbb{R})$ induces

$$\psi_M : \mathbb{RP}^n \longrightarrow \mathbb{RP}^n \text{ given by } l_a \mapsto l_{Ma}$$

$$\psi_M^* : \mathbb{RP}^{n*} \longrightarrow \mathbb{RP}^{n*} \text{ given by } l_b \mapsto l_{bM}$$

$$a \in \mathfrak{h}_{\psi_M^* b} \Leftrightarrow b \in \mathfrak{h}_{\psi_M a}^* \Leftrightarrow bMa = 0 \Leftrightarrow \psi_M a \in \mathfrak{h}_b$$

$\mathfrak{h}_{\psi_M^* b} = \psi_{M^{-1}}(\mathfrak{h}_b)$.

$E \subset \mathbb{RP}^n$

$$E^o = \text{polar of } E \equiv \{b \in \mathbb{RP}^{n*} \mid \mathfrak{h}_b \subset E^c\} = \{b \in \mathbb{RP}^{n*} \mid ba \neq 0, \forall a \in E\} = \left(\cup_{a \in E} \mathfrak{h}_a^*\right)^c$$

$(\psi_M E)^o = \psi_{M^{-1}}^*(E^o)$, $E_1 \subset E_2 \Rightarrow E_1^o \subset E_2^o$, $(\cup E_j)^o = \cap E_j^o$. If E is closed/open, then E^o is open/closed (respectively).

$$E^{oo} = (\cup_{b \in E^o} \mathfrak{h}_b)^c = (\text{union of all hyperplanes in } E^c)^c$$

So $E \subset E^{oo}$.
Projective Hahn-Banach *(Version 2)* E (APS-)convex and open/closed $\Rightarrow E = E^{oo}$.
Standard Affinization for $\mathbb{RP}^n$:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n \leftrightarrow \begin{pmatrix} 1 \\ a_1 \\ \cdot\cdot \\ \vdots \\ \cdot\cdot \\ a_n \end{pmatrix} \in \mathbb{RP}^{n+1}$$

3 useful affinizations for $\mathbb{RP}^{n*}$:
(1). $(b_1, \ldots, b_n) \in \mathbb{R}^{n*} \leftrightarrow (1 : -b_1 : \cdots : -b_n) \in \mathbb{RP}^{n+1}$, $\mathfrak{h}_b = \{a \in \mathbb{R}^n \mid \sum_{j=1}^n a_j b_j = 1\}$. Then $\mathbb{R}^{n*} \leftrightarrow$ all affine hyperplanes not passing through 0. Define $f_b(a) = ba$.

$$E \subset \mathbb{R}^n \Rightarrow E^o = \{b \in \mathbb{R}^{n*} \mid f_b \neq 1 \text{ on } E\}$$

So by defining E^o like this, $0 \in E^o$, $0 \in E$ connected implies

$$E^o = \{b \in \mathbb{R}^{n*} \mid f_b < 1 \text{ on } E\} \text{ and } E^{oo} = \{a \in \mathbb{R}^n \mid f_b(a) < 1, \forall b \in E^o\} \supset E \text{ convex}$$

If E is convex and $0 \in E$, then $E^{oo} = E$.

Exercise 2.33. If E is a connected set and $0 \in E$, then E^{oo} is the smallest convex set containing E.

E open unit ball for some Banach norm on $\mathbb{R}^n$ $\Leftrightarrow$ $E \ni 0$ open, bounded, convex and $E = -E$.
This implies that E^o is a closed unit ball for the dual norm. By Hahn-Banach, the double dual norm is the same as the original norm.

Example 2.34. Let $1 < p < \infty$.

$$E = \{\sum |a_j|^p < 11\} \Rightarrow E^o = \{\sum |b_j|^{\frac{p}{p-1}} \leq 1\}$$

More standard: $E^o = \{b \in \mathbb{R}^{n*} \mid f_b \leq 1 \text{ on } E\}$.
$M \in \mathrm{GL}(n, \mathbb{R}) \Rightarrow (ME)^o = (E^o)^{M^{-1}}$.

Exercise 2.35. Let $T : \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \mapsto \begin{pmatrix} a_1 + 1 \\ a_2 \end{pmatrix}$ and $F : (b_1, b_2) \mapsto (\frac{b_1}{1+b_1}, \frac{b_2}{1+b_1})$. Then $(TE)^o = F(E^o)$.

(2). This case is for $n = 1$. $b \in \mathbb{R}^* \leftrightarrow (-b : 1) \in \mathbb{RP}^{1*}$. So $(-b, 1) \cdot \begin{pmatrix} 1 \\ a \end{pmatrix} = 0 \Leftrightarrow -b + a = 0 \Leftrightarrow a = b$. Therefore $\mathfrak{h}_b = \{b\}$. So $E \subset \mathbb{R} \Rightarrow E^o = \mathbb{R} \setminus E$ and $E^{oo} = E$ always.
(3). Identify $b = (b_1, \ldots, b_n) \in \mathbb{R}^{n*}$ with $(b_n : -b_1 : \cdots : -b_{n-1}; 1) \in \mathbb{RP}^{n*}$.

$$\mathfrak{h}_b = \{a \in \mathbb{R}^n \mid \sum_{j=1}^{n-1} a_j b_j = a_n + b_n\} = \{a \in \mathbb{R}^n \mid a_n = \sum_{j=1}^{n-1} a_j b_j - b_n\}$$

So $\mathbb{R}^{n*} \leftrightarrow$ non-vertical affine hyperplanes in $\mathbb{R}^n$. This is useful when studying graphs of functions on $\mathbb{R}^{n-1}$.

Definition 2.36. $\mathbb{R}^{n-1} \xrightarrow{f} \mathbb{R}$ is **convex** if epigraph$(f) \equiv \{a \in \mathbb{R}^n \mid a_n \geq f(a_1, \ldots, a_{n-1})\}$ is convex.

$$\begin{aligned}(\text{epigraph}(f))^o &= \{b \in \mathbb{R}^{n*} \mid \sum_{j=1}^{n-1} a_j b_j < f(a_1, \ldots, a_{n-1}) + b_n, \forall \vec{a} \in \mathbb{R}^{n-1}\} \\ &= \left\{ b \in \mathbb{R}^{n*} \mid b_n > \sup_{\vec{a} \in \mathbb{R}^{n-1}} \{\sum a_j b_j - f(a_1, \ldots, a_{n-1})\} \right\} \\ &= \left\{ b \in \mathbb{R}^{n*} \mid b_n > f^*(b_1, \ldots, b_{n-1}) \right\}\end{aligned}$$

where $f^*(b_1, \ldots, b_{n-1})$ is the **Legendre transform** of f and is defined as:

$$f^*(b_1, \ldots, b_{n-1}) = \sup_{\vec{a} \in \mathbb{R}^{n-1}} \{\sum a_j b_j - f(a_1, \ldots, a_{n-1})\}$$

Exercise 2.37. Use Hahn-Banach to show that if f is convex, then $f^{**} = f$. Explain why exclusion of vertical hyperplane doesn't cause trouble.

$\mathbb{R}^{n*} \leftrightarrow$ all affine hyperplanes in $\mathbb{R}^n = \mathbb{RP}^{n*} \setminus \{\text{point}\}$. Not homeomorphic unless $n = 1$.

Lecture 11. October 2, 2009

If $E \subset \mathbb{R}^n$, then let $E^o = \{b \in \mathbb{R}^{n*} \mid \sum_{j=0}^{n-1} a_j b_j \neq a_n + b_n, \forall a \in E\}$.
Problem (unsolved): Which E are equivalent (via LFT/affine maps on a global/local scale) are equivalent to E^o?
See Barvinok, "Course on Convexity" (page 147) for more information.
Work with $\begin{pmatrix} a \\ a_n \end{pmatrix} \in \mathbb{R}^n$ and $(b, b_n) \in \mathbb{R}^{n*}$, where $a \in \mathbb{R}^{n-1}$ and $b \in \mathbb{R}^{n-1}$. Let $\mathbb{R}^{n-1} \xrightarrow{f} \mathbb{R} \cup \{\infty\}$. The epigraph of f is: $\{a_n \geq f(a)\}^o = \{b_n > f^*(b)\}$, where $f^*(b) = \sup_{a \in \mathbb{R}^{n-1}}(ba - f(a))$.

$$\begin{aligned} f \text{ convex } &\Leftrightarrow \{a_n \geq f(a)\} \text{ convex} \\ &\Rightarrow \{a_n \geq f_n\}^{oo} = \{a_n \geq f(a)\} \text{ by Hahn-Banach} \\ &\Leftrightarrow f^{**} = f \end{aligned}$$

$f^*(b) \geq ba - f(a) \Rightarrow f(a) \geq ba - f^*(b)$

Example 2.38. Let $f(a) = \frac{|a|^p}{p}, p > 1$ and $f^*(b) = \frac{|b|^q}{q}, q = \frac{p}{p-1}$.
$f^*(b) = \sup_{a \in \mathbb{R}^{n-1}}(ba - f(a)) = \sup_{a \in \mathbb{R}^{n-1}} ba - \frac{|a|^p}{p}$
$\frac{d}{dx} bz - \frac{z^p}{p} = b - z^{p-1} = 0 \Leftrightarrow z = b^{\frac{1}{p-1}}$
Then $f^*(b) = bb^{\frac{1}{p-1}} - \frac{|b^{\frac{1}{p-1}}|^p}{p} = b^{\frac{p}{p-1}} - \frac{|b|^{\frac{p}{p-1}}}{p} = b^q - \frac{|b|^q}{p} = \frac{|b|^q}{q}$

Example 2.39. $f(a) = |a|$ and $f^*(b) = \begin{cases} 0 & \text{if } |b| \leq 1 \\ \infty & \text{if } |b| > 1 \end{cases}$.

Now focus on "nice" situations: Assume that:
(1) $f^*(b) < \infty$
(2) f convex.
(3) f is $C^1 \Rightarrow f^*(b) = ba - f(a)$, where a is the solution to $f'(a) = b$
(4) f is $C^2 \Rightarrow \begin{pmatrix} \frac{\partial^2 f}{\partial a_j \partial a_k} \end{pmatrix} > 0$

By the inverse function theorem, a is a C^1 function of b.

$$\text{Recall: } ba - f^*(b) \text{ is } \begin{cases} \leq f(a) & \forall a, b \\ = f(a) & \text{for } a, b \text{ related as above} \end{cases}$$

Hence $(f^*)'(b) = a$. All together, $f(a) + f^*(b) = ba, f'(a) = b, (f^*)'(b) = a$. For "unrelated" a, b we still have $ba \leq f(a) + f^*(b)$.

Example 2.40. For $1 < p < \infty$, $f(a) = \frac{||a||_p}{p}$ and $f^*(b) = \frac{||b||_q}{q}$, where $q = \frac{p}{p-1}$.

$$ba \leq \frac{||a||_p}{p} + \frac{||b||_q}{q} \leq ||a||_p, \quad \text{if } ||a||_p = ||b||_q$$

Exercise 2.41. Rescale to get Holder's inequality $ba \leq ||a||_p ||b||_q$.

Definition 2.42. Let $E \subset \mathbb{RP}^n$. Then E is **$\mathbb{R}$-linearly convex** if E^c is a union of projective hyperplanes. Equivalently, $E^{oo} = E$.

Note that linearly is sometimes replaced with lineally (same meaning).
(APS)-convex implies:

- projectively convex, but not vice versa (for instance consider $\mathbb{RP}^n \setminus \{\text{point}\}$.
- $\mathbb{R}$-linearly convex (Hahn-Banach), but not vice versa (for instance consider 2 points)

Projectively convex does not imply $\mathbb{R}$-linearly convex (consider $\mathbb{RP}^n \setminus \{\text{point}\}$) and $\mathbb{R}$-linearly convex does not imply projectively convex (consider 2 points).

Proposition 2.43. *Let E be an $\mathbb{R}$-linearly convex, connected proper subset of $\mathbb{RP}^n$. Then E is convex.*

Proof. $\exists$ hyperplane in E^c. After a LFT, $E \subset \mathbb{R}^n$. So $a \in E^c \Rightarrow \exists$ hyperplane H through a such that $H \subset E^c$. Since E is connected, E lies on one side of H. Therefore E is an intersection of half-spaces and so E is convex. ☐

Can also show: $E \subsetneq \mathbb{RP}^n$ $\mathbb{R}$-linearly convex, then each component of E is convex.

Lecture 12. October 5, 2009

$f^*(b) = \sup_a (b \cdot a - f(a))$
Why?

- polarity
- inequalities (i.e. $b \cdot a \leq f^*(b) + f(a)$)
- useful for Hamiltonian mechanics (see Arnold's "Math Methods in Classical Mechanics")
- other areas of physics
- Fourier analysis: the Fourier transform is $\hat{g} = \int_{\mathbb{R}^n} e^{ix \cdot t} g(x) dx$ for $x, t \in \mathbb{R}^n$.
  If $g \in L^1 \Rightarrow g$ is continuous and $||\hat{g}||_\infty \leq ||g||_1$
  If $g \in L^2 \Rightarrow \hat{g} \in L^2$
  Suppose $\mathbb{R}^n \xrightarrow{f} \mathbb{R} \cup \{\infty\}$ is convex and $||e^f g||_1 \leq 1$ for some g.
  $$||e^{-ix \cdot (t+is)} g(x)|| = e^{x \cdot s} |g(x)| \leq e^{f^*(s)} e^{f(x)} |g(x)|$$
  $|\hat{g}(t + is)|$ iis deifned and $\leq e^{f^*(s)}$ when $f^*(s) < \infty$. $\hat{g}(t + is)$ is defined on $A \equiv \mathbb{R}^n \times i\{s \in \mathbb{R}^n \mid f^*(s) < \infty\}$. *Exercise:* $\hat{g}$ is holomorphic on the interior of A.
- see wikipedia for more

2.1. $\mathbb{CP}^n$.

Definition 2.44. Let $E \subset \mathbb{CP}^n$. E is **$\mathbb{C}$-linearly convex** if $E^c = \mathbb{CP}^n \setminus E$ is a union of $\mathbb{C}$-projective hyperplanes.

Example 2.45 (Examples of $\mathbb{C}$-linearly convex sets).

- $n = 1 \Rightarrow$ all E are $\mathbb{C}$-linearly convex
- $E \subset \mathbb{C}^n \Rightarrow \Big(E$ is $\mathbb{C}$-linearly convex $\Leftrightarrow \mathbb{C}^n \setminus E$ is a union of $\mathbb{C}$-affine hyperplanes$\Big)$
- $\{E_j\}$ are $\mathbb{C}$-linearly convex $\Rightarrow \cap E_j$ is $\mathbb{C}$-linearly convex
- $E_1 \subset \mathbb{C}^{n_1}, E_2 \subset \mathbb{C}^{n_2}$ are $\mathbb{C}$-linearly convex $\Rightarrow E_1 \times E_2$ is $\mathbb{C}$-linearly convex
- $E \subset \mathbb{CP}^n$ is $\mathbb{C}$-linearly convex $\Rightarrow \text{Int}E$ is $\mathbb{C}$-linearly convex
- $E \subset \mathbb{CP}^n$ is $\mathbb{C}$-linearly convex does not imply $\overline{E}$ is $\mathbb{C}$-linearly convex

Proposition 2.46. *Let l be a $\mathbb{C}$-affine line and $l \subset E \subset \mathbb{C}^n$, where E is $\mathbb{C}$-linearly convex. Then $E \cong^{\text{affine}} \mathbb{C} \times E'$.*

Proof. Assume that l is the z_1-axis. Let $H \subset E^c$ be a $\mathbb{C}$-hyperplane. Then H par. to z_1 axis. E^c is union of hyperplanes par. to z_1-axis. Therefore $E^c = \mathbb{C} \times G \Rightarrow E = \mathbb{C} \times E'$. ☐

Lemma 2.47. *If E is open and $\mathbb{C}$-linearly convex, then E is pseudo-convex. However, the converse is not true.*

Proof. $E = \cap_{H \subset E^c \text{ hyperplane}} (\mathbb{CP}^n \setminus H)$ ☐

Example 2.48. Let $E \subset \mathbb{C}^n$ be $\mathbb{C}$-linearly convex and $Q : \mathbb{C}^n \to \mathbb{C}^{n-1}$ projection. This does not imply that $Q(E)$ is $\mathbb{C}$-linearly convex.

Setting up the definitions of projectively $\mathbb{C}$*-convex and* $\mathbb{C}$*-convex.*
Let $E \subset \mathbb{CP}^1 =$ Riemann Sphere.
E is projectively $\mathbb{C}$-convex $\Leftrightarrow E, E^c$ are connected $\Leftrightarrow E^c$ projectively $\mathbb{C}$-convex
E is $\mathbb{C}$-convex $\Leftrightarrow E$ is projectively $\mathbb{C}$-convex and $E \neq \mathbb{CP}^1$
$E \subset \mathbb{CP}^1$ open $\Rightarrow \big(E$ is projectively $\mathbb{C}$-convex $\Leftrightarrow E$ connected and simply connected $\big)$
$E \subset \mathbb{CP}^1$ is open or closed and bE is a smooth manifold without boundary $\Rightarrow \big(E$ is $\mathbb{C}$-convex $\Leftrightarrow E = \emptyset$ or bE is one simple closed curve $\big)$
$E \subset \mathbb{C}$ is $\mathbb{C}$-convex $\Leftrightarrow \mathbb{C} \setminus E$ has no boundary components.

Example 2.49. If $E_1, E_2 \subset \mathbb{C}$ are $\mathbb{C}$-convex, this does not imply that $E_1 \cap E_2$ is $\mathbb{C}$-convex.

Lecture 13. October 7, 2009

Definition 2.50. Let $E \subset \mathbb{CP}^n$. E is **projectively** $\mathbb{C}$**-convex** if $l \cap E$ and $l \setminus E$ are conncected for all projective $\mathbb{C}$-lines l. E is $\mathbb{C}$**-convex** if E is projectively convex and E contains no projective $\mathbb{C}$-line. E is $\mathbb{C}$**-linearly convex** if $\mathbb{CP}^n \setminus E$ is a union of projective $\mathbb{C}$-hyperplanes.

All of these are invariant under LFTs.

Definition 2.51. Let $E \subset \mathbb{C}^n$. E is $\mathbb{C}$**-convex** if E is projectively $\mathbb{C}$-convex. Equivalently, $l \cap E$ is connected and $l \setminus E$ has no bounded components for affine $\mathbb{C}$-line l. E is $\mathbb{C}$**-linearly convex** if $C^n \setminus E$ is a union of affine $\mathbb{C}$-hyperplanes.

Example 2.52. Given $E \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{CP}^1$, then E is $\mathbb{C}$-convex $\Leftrightarrow E$ is connected.

Exercise 2.53. Intersections of any affine $\mathbb{C}$-line l with $\mathbb{R}^n \subset \mathbb{C}^n$ are empty, a point or an affine $\mathbb{R}$-line.

Corollary 2.54. *Given $E \subset \mathbb{R}^n \subset \mathbb{C}^n \subset \mathbb{CP}^n$, then E is $\mathbb{C}$-convex $\Leftrightarrow E$ is $\mathbb{R}$-convex.*

Exercise 2.55. Intersections of affine $\mathbb{C}$-hyperplanes with $\mathbb{R}^n$ are empty, affine $\mathbb{R}$-hyperplanes, or $(n-2)$-dimensional affine $\mathbb{R}$-planes.

Corollary 2.56. *Given $E \subset \mathbb{R}^n$, then E is $\mathbb{C}$-linearly convex $\Leftrightarrow \mathbb{R}^n \setminus E$ is a union of $(n-2)$-dimensional affine $\mathbb{R}$-planes.*

Example 2.57. Let $\Delta \subset \mathbb{C}^n$ be the open unit disk. $\Delta \times \Delta, \overline{\Delta} \times \overline{\Delta}$ are $\mathbb{C}$-convex.

Theorem 2.58. *Let $E_1 \subset \mathbb{C}^{n_1}, E_2 \subset \mathbb{C}^{n_2}$ both open or both compact, but neither a point, nor empty, nor all of $\mathbb{C}^{n_j}$. If $E_1 \times E_2$ is $\mathbb{C}$-convex, then E_1, E_2 are $\mathbb{R}$-convex.*

Proof. See APS, Prop. 2.2.5. □

We will show that if $E \subset \mathbb{CP}^n$ is $\mathbb{C}$-convex and open/closed, then E is $\mathbb{C}$-linearly convex. This is the complex projective version of the Hahn Banach theorem. However, if E is $\mathbb{C}$-linearly convex, this does not imply that E is $\mathbb{C}$-convex. We will also show that if E is $\mathbb{C}$-linearly convex and open/closed with C^1 boundary, then E is $\mathbb{C}$-convex.
What does C^1 boundary mean? If E is compact, this means that E is a $2n$-dimensional manifold with boundary and bE is a $(2n-1)$-dimensional manifold without boundary.

$$\mathbb{CP}^n = (\mathbb{C}_{\text{col.}} \setminus \{0\})/\sim \text{ and } \mathbb{CP}^{n*} = (\mathbb{C}_{\text{row}} \setminus \{0\})/\sim$$

$$\begin{pmatrix} a_0 \\ \cdot\cdot \\ \vdots \\ \cdot\cdot \\ a_n \end{pmatrix} \in \mathbb{CP}^n \text{ and } (b_0 : \cdots : b_n) \in \mathbb{CP}^{n*}$$

$$a \in h_b \Leftrightarrow b \in h_a^* \Leftrightarrow ba = \sum_{j=0}^{n} a_j b_j = 0$$

$\mathbb{CP}^{n*}$ is the set of $\mathbb{C}$-hyperplanes in $\mathbb{CP}^n$ and h_a^* is the set of $\mathbb{C}$-hyperplanes through a.

Definition 2.59. Let $E \subset \mathbb{CP}^n$. The **dual complement** E^* of E is:

$$E^* = \{b \in \mathbb{CP}^{n*} \mid h_b \subset E^c\} = \{b \in \mathbb{CP}^{n*} \mid ba \neq 0, \forall a \in E\} = \left(\cup_{a\in E} h_a^*\right)^c$$

As before, $(\psi_M E)^* = \psi^*_{M^{-1}}(E^*)$, where $\psi^*_{M^{-1}} : l_b \to l_{bM^{-1}}$. If $E_1 \subset E_2$, then $E_1^* \supset E_2^*$. If E is open (closed), then E^* is closed (open). In addition, $(\cup E_j)^* = \cap E_j^*$.

$$E \subset E^{**} \text{ always and } E = E^{**} \Leftrightarrow E \text{ is } \mathbb{C}\text{-linearly convex}$$

E^* is always $\mathbb{C}$-linearly convex and $(E^*)^c = \cup_{a\in E} h_a^*$ so E^{**} is always $\mathbb{C}$-linearly convex.

If $E \subset F$ and F is $\mathbb{C}$-linearly convex, then $E^* \supset F^* \Rightarrow E^{**} \subset F^* * = F$. So E^{**} is the smallest $\mathbb{C}$-linerly convex set containing E (i.e. it is the **$\mathbb{C}$-linear convex hull** of E).

Proposition 2.60. *If E is $\mathbb{C}$-linearly convex, then $IntE$ is $\mathbb{C}$-linearly convex.*

Proof. Int$E \subset (\text{Int}E)^{**} \subset E^{**} = E$ and Int$E)^{**}$ is open so $E = \text{Int}E)^{**}$. □

Theorem 2.61. *Suppose $E \subset \mathbb{C}^n$ is compact and $\mathbb{C}$-linearly convex, E^* is connected, and $a \notin E$. Then there exists a polynomial p such that $|p(a)| > \max_E |p|$.*

Lecture 14. October 9, 2009

Definition 2.62. Let $E \subset C^m$ compact. Then E is **polynomial convex** if for $a \notin E, \exists$ a polynomial p such that $|p(a)| > \max_E |p|$.

Theorem 2.63. *If E is polynomial convex, then all functions holomorphic on a neighborhood of E are E-uniform limits of polynomials.*

Theorem 2.64. *Let $E \subset \mathbb{C}^m$ be compact and $\mathbb{C}$-linearly convex. If E^* is connected, then E is polynomial convex.*

Proof. Suppose $a \notin E$. Choose $f : [0,1] \to E^*$ with $h_{f(a)}$ = hyperplane at ∞, $a \in h_{f(a)}$. $h_{f(t)} = \{g_t = 0\}$, where g_t is a 1st degree polynomial which depends continuously on $t > 0$. Let $S = \{t \in (0,1) \mid \frac{1}{g_t}$ is $(E \cup \{a\}) -$ uniform limit of polynmoials$\}$. Then $(0, \epsilon) \subset S, S$ is closed and S is open (for $t_0 \in S$ with $t_0 \approx t$, $\frac{1}{g_{t_0}} \sum_{j=0}^{\infty} \left(1 - \frac{g_t}{g_{t_0}}\right)^j = \frac{1}{g_{t_0}} \frac{1}{\frac{g_t}{g_{t_0}}} = \frac{1}{g_t}$ so $(E \cup \{\alpha\})$-uniform limit of polynomials $\Rightarrow t \in S$). Therefore $S = (0,1)$. $\left|\frac{1}{g_{1-\epsilon}(\alpha)}\right| > \max_E \left|\frac{1}{g_{t-\epsilon}}\right|$. Approximate polynomials on $E \cup \{\alpha\}$ and get $|p(a)| > \max_E |p|$. □

Remarks:

- Just need to pull some hyperplane through a to ∞ avoiding E
- E is polynomial convex $\Rightarrow E$ is $\mathbb{C}$-linearly convex.
  Example: $\{(z_1, z_2) \mid z_2 = z_1^2, |z_1| \leq 1\}$ *is polynomial conve. E is not $\mathbb{C}$-linearly convex.*
- *(Stalzenberg, 1963)* E is polynomial convex $\Leftrightarrow$ all $a \notin E$ lie in an algebraic hyperplane that can be pulled to ∞ avoiding E.

Projective $\mathbb{R}$-planes in $\mathbb{RP}^n$:

- They are closed submanifolds
- They are flat with respect to any affinizations (equivalently, $\mathbb{R}$-LFTs map affine $\mathbb{R}$-planes to $\mathbb{R}$-affine planes)

These properties also hold in $\mathbb{CP}^n$.
What about $\mathbb{R}$-projective planes in $\mathbb{CP}^n$?
Example: $\mathbb{R}\cup\{\infty\}\subset\mathbb{CP}^1=\mathbb{C}\cup\{\infty\}$ is a closed submanifold byt it is not flat with respect to any affiniztaions (real line maps to a circle).

Exercise 2.65. Let E be an affine $\mathbb{R}$-plane in $\mathbb{C}^n\cong\mathbb{R}^{2n}$. Let F be the closure of E in $\mathbb{CP}^n$.
(1) F is a manifold $\Leftrightarrow$ E complex or totally real
(2) F is flat with respect to all affinizations $\Leftrightarrow$ E is complex

Let $E\subset\mathbb{C}^n\cong\mathbb{R}^{2n}$ so that $E^*\subset\mathbb{CP}^{n*}$ and $E^o\subset\mathbb{RP}^{2n*}$. 3 affinizations for E^* are:

3: ($\sim$ Legendre transform) - discuss later

2: For $n=1, b\in\mathbb{C}\leftrightarrow[-b:1]\in\mathbb{CP}^*$ and $a\in\mathbb{C}\leftrightarrow\begin{pmatrix}1\\ ..\\ a\end{pmatrix}\in\mathbb{CP}^1$. Then $h_a^*=\{b\mid -b+a=0\}=\{a\}$ and $E^*=\mathbb{C}\setminus E$.

1:

$$a\in\mathbb{C}^n\leftrightarrow\begin{pmatrix}1\\ ..\\ a_1\\ ..\\ \vdots\\ ..\\ a_n\end{pmatrix}\in\mathbb{CP}^n \text{ and } b\in\mathbb{C}^{n*}\leftrightarrow[1:-b_1:\cdots:-b_n]\in\mathbb{CP}^{n*}$$

$h_b^*=\{a\in\mathbb{C}^n\mid\sum_{j=0}^{\infty}a_jb_j=1\}$
The real dot product of a and b corresponds to the real part of the complex dot product of a and $\bar{b}$.

$$h_b^{*\mathbb{R}}=\{a\mid \mathrm{Re}\bar{b}a=1 \text{ and } h_b^{*\mathbb{C}}=\{a\mid ba=1\}$$

For $n=1$,

$$h_b^{\mathbb{C}}=\{\frac{1}{b}\} \text{ and } h_b^{\mathbb{R}}=\{\frac{1_i t}{\bar{b}}\mid t\in\mathbb{R}\}$$

So $h_{\bar{b}}^{\mathbb{C}}$ is the point on $h_b^{\mathbb{R}}$ closest to 0.
For $n>1$,
$h_b^{\mathbb{R}}$ is a disjoint union of $\mathbb{C}$-hyperplanes and $h_{\bar{b}}^{\mathbb{C}}$ is a $\mathbb{C}$-hyperplane in $h_b^{\mathbb{R}}$ closest to 0.
If $f_b: a\mapsto ba$, then:

$$b\in E^*\Leftrightarrow 1\notin f_b(E) \text{ and } b\in E^o\Leftrightarrow 1\notin \mathrm{Re}f_{\bar{b}}(E)$$

$E^o\subset$ conjugate of E^*.

Lecture 15. October 14, 2009

$E^o,^*$ affine version (1) with $a\in\mathbb{C}^n, b\in\mathbb{C}^{n*}$ and $f_b:\mathbb{C}^n\to\mathbb{C}$ by $a\mapsto ba$. Suppose $E\subset\mathbb{C}^n$. Then

$$b\in E^*\Leftrightarrow 1\notin f_b(E)$$
$$b\in E^o\Leftrightarrow 1\notin \mathrm{Re}F_{\bar{b}}(E)$$

So E^* lines in conj(E^o).

Definition 2.66. E is **circular** if for $a\in E,\theta\in\mathbb{R}\Rightarrow e^{i\theta}a\in E$. E is **complete circular** if for $a\in E,|\lambda|\le 1\Rightarrow\lambda a\in E$.

If E is circular and $b\in\mathbb{C}^{n*}$, then

- $f_b(E)$ is a disk centered at 0 or all of $\mathbb{C}$

- $E^* = \text{conj}(E^o)$
- if $E = \text{conj}(E)$, then $E^* = E^o$

Definition 2.67. E is a **Reinhardt** domain if $(a_1, \dots, a_n) \in E$ and $\theta_1, \dots, \theta_n \in \mathbb{R}$, then $(e^{i\theta_1}a_1, \dots, e6i\theta_n a_n) in E$. E is a **complete Reinhardt** domain $(a_1, \dots, a_n) \in E, |\lambda_j| \leq 1$, then $(\lambda_1 a_1, \dots, \lambda_n a_n) \in E$.

E complete Reinhardt $\Rightarrow E$ complete circular, $E = \text{conj}(E) \Rightarrow E^* = E^o$.

Example 2.68. If $E = \{\sum_j |a_j|^p \leq 1\}, 1 < p < \infty$, then $E^o = E^* = \{\sum_j |b_j||^q < 1\}, q = \frac{p-1}{p}$.

Proposition 2.69. *If $E \subset \mathbb{C}^n$ is open and $\mathbb{C}$-convex, then E is connected and simply connected.*

The converse is not true.

Proof. Any two points in E have a line connecting them in E so E is path connected $\Rightarrow$ connected. Given $\gamma : [0,1] \to E, \gamma(0) = \gamma(1) = a$. Need $h : [0,1]^2 \to E$. Let S be a square with $\gamma(t)$ along the left edge and a along the rest of the edges with t as the variable from bottom to top and s is the variable from left to right. E is open and $[0,1]$ is compact, so we can partition S into finitely many open sets. Partition the left edge as: $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = 1$ and $h_j = (t_j - t_{j-1}) \times [0,1]$. $h_j(s,t)$ is defined on h_j and $h_j(s,t)$ is contained in a $\mathbb{C}$-line for fixed t. We want to extend our definition of the h_j to one map h, but we don't know that $h_j(s,t_j) = h_j(s,t_{j-1})$. However, they are homotopic within $E \cap$ (line). Use this to assemble h. □

Theorem 2.70. *If $E \subset \mathbb{C}^2$ is open and $\mathbb{C}$-convex, then E is $\mathbb{C}$-linearly convex.*

Proof. It suffices to prove that it $0 \in E^c$, then $0 \in l \subset E^c$, where l is a complex line. Suppose not. For each $\zeta \in \mathbb{C}$ (a slope), let $E_\zeta = \{z \mid (z, \zeta z) \in E\}$. The E_ζ are open, connected, simply-connected and non-empty with $0 \notin E_\zeta$. We can choose a continuous branch of $\arg z$ on E_ζ, in particular $\arg_\zeta(z)$ (determined up to $2\pi\mathbb{Z}$). The set of all possible choices forms a $\mathbb{Z}$-bundle over $\mathbb{C}$ and a covering spaces over $\mathbb{C}$. We can choose $\arg_\zeta(z)$ to be continuous in ζ. We have been ignoring the vertical line in this so far, so let's fix that. Let $\widetilde{E}_\zeta = \{z \mid (\zeta z, z) \in E\}$. In the same way, we get $\widetilde{\arg_\zeta}(z)$. How do these relate?

$$\text{Pick } \zeta \in \mathbb{C} \setminus \{0\} \Rightarrow (z \in \widetilde{E}_\zeta \Leftrightarrow \zeta z \in E_{\frac{1}{\zeta}}$$

$$z \in \widetilde{E}_\zeta \Rightarrow -\widetilde{\arg_\zeta}(z) + \arg_{\frac{1}{\zeta}}(\zeta z) = \arg(\zeta)$$

This gives a continuous branch of the arg on $\mathbb{C} \setminus \{0\}$, which is a contradiction since there is no continuous branch of arg on $\mathbb{C} \setminus \{0\}$. □

Proposition 2.71. *Let $V \subset \mathbb{C}^n$ be an affine $\mathbb{C}$-plane and $E \subset \mathbb{C}^n$ be $\mathbb{C}$-convex. Then $E \cap V$ is $\mathbb{C}$-convex.*

Proposition 2.72. *Suppose $E \subset \mathbb{C}^n$ is open, $\mathbb{C}$-convex, V is an affine $\mathbb{C}$-plane, and $Q : \mathbb{C}^n \to V$ is an affine projection. Then $Q(E)$ is $\mathbb{C}$-convex.*

Theorem 2.73. *If $E \subset \mathbb{C}^n$ is open and $\mathbb{C}$-convex, then E is $\mathbb{C}$-linearly convex.*

Proof. By induction on n assuming the previous proposition. Let $a \in E^c$. When $n = 1$ this is clear and we already proved the case when $n = 2$. By the $n = 2$ case, $\exists a \in l \subset E^c$ line. Choose $Q : \mathbb{C}^n \to V$, where V is an affine hyperplane and Q collapses l to a point. Since $a \in l \subset E^c, Q(l) \notin Q(E) \subset V$ (note: $Q(E) \subset V$ is $\mathbb{C}$-convex). Get $Q(l) \subset W \subset V \setminus Q(E)$, where $\dim W = n - 2$. Get $a \in Q^{-1}(W) \subset E^c$ with dimension $n - 1$. □

Lecture 16. October 16, 2009

Last time:

Theorem 2.74. *If $E \subset \mathbb{C}^n$ is open and $\mathbb{C}$-convex, then E is $\mathbb{C}$-linearly convex.*

The proof used proposition 1:

Proposition 2.75 (1)**.** *$Q : \mathbb{C}^n \to V$ affine projection and $E \subset \mathbb{C}^n$ open, $\mathbb{C}$-convex, then $Q(E)$ is $\mathbb{C}$-convex.*

Proposition 2.76 (2)**.** *$E \subset \mathbb{C}^n$ is open and $\mathbb{C}$-convex, then E is connected and simply-connected.*

Proposition 2.77 (3)**.** *If $V \subset \mathbb{C}^n$ is an affine plane and $E \subset \mathbb{C}^n$ is $\mathbb{C}$-convex, then $V \cap E$ is $\mathbb{C}$-convex.*

Proof of Proposition 1. Claim 1: $Q(E)$ is connected
Claim 2: $Q(E)$ is simply-connected: take a loop $\gamma \subset Q(E)$ and break it into arcs which when lifted to $\mathbb{C}^n$ are arcs in E. Then connected the endpoints of these arcs in apropriate ways. We can do this so that the loop we get is inside E since E is $\mathbb{C}$-convex, so we get a loop $\widetilde{\gamma}$ such that $Q(\widetilde{\gamma})$ is homotopic to γ. By proposition 2, $\widetilde{\gamma}$ $\sim$point in E. Project the homotopy: then $\gamma \sim$ point in $Q(E)$. Now consider a line $l \subset V$. $E \cap Q^{-1}(l)$ is $\mathbb{C}$-convex. Claims 1 and 2 imply that $Q(E \cap Q^{-1}(l)) = Q(E) \cap l$ is connected and simply-connected. □

Projections in Projective Space:
Given $a \notin H \subset \mathbb{CP}^n$, where H is a hyperplane, we get $Q : \mathbb{CP}^n \setminus \{a\} \to H$ such that for $b \in \mathbb{CP}^n \setminus \{a\}$, $b \mapsto \vec{ab} \cap H$, where $\vec{ab} \cap H$ is the point given by the line connecting a and b that intersects H.

Proposition 2.78 (4)**.** *For $n = 2$ and a, H, Q as above, $a \notin E$, where E is an open, $\mathbb{C}$-convex set. Then $Q(E) \neq H$.*

Proof of Proposition 4. We may assume that $a = 0 \in \mathbb{C}^2 \subsetneq \mathbb{CP}^2$ and H =the line at $\infty = \mathbb{CP}^1$. Then $Q : (z_1, z_2) \mapsto (z_1 : z_2)$. Suppose to the contrary that $Q(E) = \mathbb{CP}^1$. ***Use Michael's theorem (1955):*** *Let X, Y be manifolds, $X \xrightarrow{\varphi} Y$ is continuous open and surjective, $\varphi^{-1}(y)$ is contractible $\forall y \in Y \Rightarrow \exists Y \xrightarrow{\psi} X$ continuous such that $\varphi \circ \psi = Id_Y$ (i.e. $\psi(y) \in \varphi^{-1}(y), \forall y \in Y$).* So there is a continuous $\psi : \mathbb{CP}^1 \to E$ such that $Q \circ \psi = \text{Id}$. Construct a "fiberwise universal cover" $E \setminus \widetilde{\psi(\mathbb{CP}^1)}$. Mimicing what we did last lecture, we get a continuous branch of

$$\arg\left(\frac{z_1\psi_1\left(\frac{z_2}{z_1}\right)}{z_1 - \psi_1\left(\frac{z_2}{z_1}\right)}\right) \text{ on } E \setminus \widetilde{\psi(\mathbb{CP}^1)} \setminus \{z_2\text{-axis}.$$

Do this again with the two variables switch. Following the same step as a proof proof from last lecture, we get a contradiction. □

Corollary 2.79. *If $E \subset \mathbb{CP}^2$ is open and $\mathbb{C}$-convex, then E is $\mathbb{C}$-linearly convex.*

Proof. By Prop. 4, $\exists$ a line in E^c. Move it to ∞ and apply the affine result. □

Addendum to Proposition 4: $Q(E)$ is $\mathbb{C}$-convex.

Proof. We may assume that E is disjoint from the z_2-axis (so $z_1 \neq 0$). Then $Q(z_1, z_2) = (z_1 : z_2) = \left(1 : \frac{z_2}{z_1}\right)$. Do a LFT change of coordinates: $w_1 = \frac{1}{z_1} w_2 = \frac{z_2}{z_1}$. New $E \subset \mathbb{C}^2$ and $Q(w_1, w_2) = w_2$. Apply the affine result from Proposition 1. □

Proposition 2.80 (5)**.** *Let $Q : \mathbb{CP}^n \setminus \{a\} \to H$ as before with $a \notin E$ and $E \subset \mathbb{CP}^n$ an open, $\mathbb{C}$-convex set. Then $Q(E)$ is $\mathbb{C}$-convex.*

Proof. By induction on n:
Let $l \subset H$ be a line. Then $Q^{-1}(l) \cup \{a\}$ is a projective 2-plane. By the "projective proposition 3", $Q^{-1}(l) \cap E$ is $\mathbb{C}$-convex. By the addendum to proposition 4, $Q(Q^{-1}(l) \cap E)) = l \cap Q(E)$ is $\mathbb{C}$-convex $\Rightarrow Q(E)$ is $\mathbb{C}$-convex. □

Corollary 2.81. *$E \subset \mathbb{CP}^n$ is open and $\mathbb{C}$-convex, then E is $\mathbb{C}$-linearly convex.*

Proof. Fix $a \notin E$. Choose a projection $Q : \mathbb{CP}^n \setminus \{a\} \to H$, where H is a hyperplane. $Q(E) \subset H$ is $\mathbb{C}$-convex. By the inductive hypothesis, $\exists W \subset H \setminus Q(E)$ of dimension $n-2$. $Q^{-1}(W) \cup \{a\}$ is a projective hyperplane in E^c that contains a. Therefore every point outside E belongs to a projective hyperplane outside of $E \Rightarrow E$ is $\mathbb{C}$-linearly convex. □

Lecture 17. October 21, 2009

Given an open set $E \subset \mathbb{CP}^1$. E is $\mathbb{C}$-convex $\Leftrightarrow$ E is connected, simply connected, and $E \neq \mathbb{CP}^1 \Leftrightarrow E$ contractible (i.e. Id: $E \to E$ homotopic to constant within E).

Theorem 2.82. *Given $E \subset \mathbb{CP}^1$ closed, non-empty. E is $\mathbb{C}$-convex $\Leftrightarrow$ E is connected, $H^1(E) = 0, H^2(E) = 0 \Leftrightarrow H^0(E) = \mathbb{R}$ and $H^k(E) = 0, \forall k > 0 \Leftrightarrow E$ has the cohomology of a point.*

$H^k(E)$ is the kth cohomology of E. If the cohomology has coefficients in $\mathbb{R}$, then E conncected is equivalent to $H^0(E) = \mathbb{R}$. $H^2(E) = 0$ ensures that E is not the Riemann sphere. Let $E = \cup_j U_j$ be a relatively open cover of E, $f_{j,k} : U_j \cap U_k \to \mathbb{R}$ locallly constant, $f_{j,k} + f_{k,l} + f_{l,j} = 0 \Rightarrow \exists f_j : U_j \to \mathbb{R}$ locally constant such that $f_{j,k} = f_j - f_k$. $f_{j,k,l} : U_j \cap U_k \cap U_l \to \mathbb{R}$ is locally constant, $f_{j,k,l} - f_{j,k,m} + f_{j,l,m} - f_{k,l,m} = 0 \Rightarrow \exists f_{j,k} : U_j \cap U_k \to \mathbb{R}$ is locally constant such that $f_{j,k,l} = f_{j,k} + f_{k,l} + f_{l,k}$.

Proof. Refer to the APS monograph. □

Proposition 2.83. *Given $E_1, E_2 \subset \mathbb{CP}^1$ both $\mathbb{C}$-convex and both open/both closed. $E_1 \cup E_2$ is $\mathbb{C}$-convex $\Leftrightarrow E_1 \cap E_2 \neq \emptyset$ is $\mathbb{C}$-convex.*

Proof. Use the Mayer-Vietoris sequence. □

Proposition 2.84 (2'). *If $E \subset \mathbb{CP}^n$ is closed and $\mathbb{C}$-convex, then E has the cohomology of a point.*

Proof. Use Mayer-Vietoris and Vietoris-Begle "blowing-up." □

Proposition 2.85 (5'). *Suppose $a \notin E \subset \mathbb{CP}^n$, where E is a closed $\mathbb{C}$-convex set. Let H be a hyperplane and $Q : E \to H$ a projection. Then $Q(E)$ is $\mathbb{C}$-convex.*

Proof. Use Vietoris-Begle Mapping Theorem. □

Theorem 2.86. *$E \subset \mathbb{CP}^n$ closed and $\mathbb{C}$-convex, then E is $\mathbb{C}$-linearly convex.*

Theorem 2.87. *Let $E \subset \mathbb{CP}^n$ be $\mathbb{C}$-convex, non-empty, and open/closed. Then E^* is $\mathbb{C}$-convex and non-empty.*

Proof. Recall that:

$$E^* = \{b \in \mathbb{CP}^{n*} \mid h_b \subset E^c\} = \{b \in \mathbb{CP}^{n*} \mid ba \neq 0, \forall a \in E\} = \left(\cup_{a \in E} h_a^*\right)^c$$

By the previous theorem, E is $\mathbb{C}$-linearly convex $\Rightarrow E^* \neq \emptyset$. $\alpha \in E \Rightarrow E^*$ disjoint from $h_\alpha^* \Rightarrow E^*$ contains no projective line. Assume this is true for dimension $n-1$. It suffices to show that $E^* \cap h_a^*$ is $\mathbb{C}$-convex, $\forall a \in \mathbb{CP}^n$.
Case 1: $E^* \cap h_a^* = \emptyset$, so nothing to prove.
Case 2: $E^* \cap h_a^* \neq \emptyset \Rightarrow$ some $b \in h_a^*$ and $h_b \subset E^c \Rightarrow ba = 0$ but $b\tilde{a} \neq 0, \forall \tilde{a} \in E \Rightarrow a \notin E$. Let H be a hyperplane and $\mathbb{CP}^n \setminus \{a\} \to H$. By proposition 5 or 5', $Q(E)$ is $\mathbb{C}$-convex.

$h_a^* =$ set of hyperplanes in $\mathbb{CP}^n$ through a and $Q : h - a^* \to$ set of hyperplanes within H (the dual of H). By induction, $Q(E)^*$ (the dual within H) is $\mathbb{C}$-convex. So $Q(E)^*$ is the set of hyperplanes in H that do not intersection $Q(E)$. Identify $Q(E)$ with $E^* \cap h_a^*$. □

Consider $\subset \mathbb{CP}^n$ open with C^1 boundary.
Given: $p \in bE \cap \mathbb{C}^n$, we have a real tangent hyperplane $p \in T_p(bE) \subset \mathbb{C}^n$. $T_p(bE)$ contains a unique $\mathbb{C}$-hyperplane $H_p(BE)$ passing through $p \to H_p(bE)$ is the unique $\mathbb{C}$-hyperplane tangent to bE at p. The T_p construction will not behave well under LFTs, but the H_p will behave well.

$$F \text{ is a LFT } \Rightarrow H_{F(p)}(b(F(E)) = F(H_p(bE))$$

In particular, also have $H_p(bE)$ for p at ∞. Now suppose that E is $\mathbb{C}$-linearly convex. Then each $p \in bE$ must lie in a $\mathbb{C}$-hyperplane $\widetilde{H} \subset E^c$ (in particular it must be the hyperplane $H_p(bE)$). Suppose $\widetilde{H} \neq H_p(bE)$.. Then $\widetilde{H}$ meets bE transversally at p. $\widetilde{H}$ has $\mathbb{R}$-dimension $2n-2$ so $\widetilde{H}$ has smooth boundary of $\mathbb{R}$-dimension $2n-3$. So $H_p(bE) \subset E^c$.

Lecture 18. October 23, 2009

General assumptions for this lecture: $E \subset \mathbb{CP}^n$ is open and bE is C^1.
Last lecture we showed:

- $p \in bE \Rightarrow \exists$ a unique projective $\mathbb{C}$-hyperplane H_p tangent to bE at p
- E is $\mathbb{C}$-linearly convex $\Rightarrow H_p \in E^c$

Assumptions that we will use at some point during this lecture:

(*): E is connected and $\forall p \in bE$ we have $p \notin \overline{H_p(bE) \cap E}$

(**): $\exists p \in bE$ such that p is isolated in $H_p(bE) \cap \overline{E}$ (or in $H_p(bE) \cap bE$)

Claim 1: $(*) \Rightarrow$ each line l meets bE transversally along $b(l \cap E)$.

Proof. If l is not transverse to bE at p, then $l \in H_p(bE) \Rightarrow p \in b(l \cap E)$. □

Claim 2: $(*) \Rightarrow l \cap E$ is connected $\forall \mathbb{C}$-lines l

Proof. Consider $p, q \in l \cap E$. Since E is connected, there is a path $\gamma : [0,1] \to E$ such that $\gamma(0) = p, \gamma(1) = q$, and $\gamma(t) \neq p$ for $t \neq 0$. Let $\Omega_t = p-$component of $E \cap ($ line through $p, \gamma(t))$. Transversality implies that $b\Omega_t$ is a union of C^1 curves varying continuously with t. Let $S = \{t \in (0,1] \mid \gamma(t) \in \Omega_t\}$. S is open in $(0,1]$, $(0,1] \setminus S$ is open in $(0,1]$, and $(0,\epsilon) \subset S$ for some $1 \leq \epsilon > 0$. Therefore $S = (0,1] \Rightarrow \gamma(1) \in \Omega_1$. □

Claim 3: $(*) \Rightarrow$ each $H_p \in E^c$
So all non-empty $l \cap E$ are connected and bounded by a fixed number, k, of C^1 curves.
Claim 4: $(*), (**) \Rightarrow k = 1$

Proof. wlog, assume that $p = 0$ is the point given by $(**)$, $T_p(bE) = \mathbb{C}^{n-1} \times \mathbb{R}$ and $H_p(bE) = \mathbb{C}^{n-1} \times \{0\}$. Locally, $E = \{\text{Im} z_n > \varphi(z_1, \ldots, z_{n-1}, \text{Re} z_n)\}$, where φ is C^1. $bE \cap (\mathbb{C}^{n-1} \times \{0\}) = \{0\} \Rightarrow \varphi(0) = 0, \varphi > 0$ at other nearby points. Let

$$\Omega_\epsilon = \{z \in \mathbb{C} \mid (z, 0, \ldots, 0, i\epsilon) \in E\} = \{z \in \mathbb{C} \mid \varphi(z, 0, \ldots, 0) < \epsilon\}$$

For $0 < \epsilon < \epsilon_0 \Rightarrow \Omega_\epsilon$ is non-empty, connected, and bounded by k-smooth curves. For $0 < \epsilon_1 < \epsilon_2 < \epsilon_0 \Rightarrow \Omega_{\epsilon_1} \subset \Omega_{\epsilon_2}$. So $\bigcap_{0<\epsilon<\epsilon_0} \Omega_\epsilon = \{0\}$. Since the boundary curves vary continuously with ϵ, it is not possible that $\bigcap_{0<\epsilon<\epsilon_0} \Omega_\epsilon$ is a point unless $k = 1$. □

Theorem 2.88. $(*), (**) \Rightarrow E$ *is* $\mathbb{C}$*-convex* $\Rightarrow E$ *is* $\mathbb{C}$*-linearly convex* $\Rightarrow (*)$.

Remarks about $(**)$*:*

(1) If $E \subset \mathbb{C}^n \mathbb{C}$
p^n bounded, then $(**)$ is automatic.

Proof. Choose $p \in E$ farthest from the origin (ties are ok). $H_p \setminus \{p\} \subset E^c$. □

(2) $(**)$ may be a consequence of $\mathbb{C}$-convexity
(3) If $E \subset \mathbb{C}^2$ strongly pseudoconvex with bEC^3, then $(**)$ fails $\Leftrightarrow E \simeq^{\text{affine}}$ (convex domain in $\mathbb{R}^2) \times i\mathbb{R}^2$ *(Result due to Bolt, 2009).* This implies that E does not have smooth boundary in $\mathbb{CP}^2$.

How do we verify condition $(*)$*?*
Reduce to the case $\mathbb{C}^{n-1} \times \mathbb{R}$. Locally $E = \{\text{Im} z_n > \varphi(z_1, \ldots, z_{n-1}, \text{Re} z_n)\}$. $(*)$ at $0 \Leftrightarrow \varphi(z_1, \ldots, z_{n-1}, 0)$ has a strict local minimum at $0 \Rightarrow$ Hessian of φ at 0 with respect to $x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}$ is ≥ 0. If the Hessian is > 0, then $(**)$ holds.

Lecture 19. October 26, 2009

Recall: $\psi : \mathbb{R}^n \to \mathbb{R}, \text{Hess}_p \psi(x) = \sum \frac{\partial^2 \psi}{\partial x_j \partial x_k}(p) x_j x_k$.
$E \subset \mathbb{CP}^n$ is open and bE is C^2.
For $p \in bE$ with $T_p(bE)$ and $H_p(bE)$ we can apply a linear transformation to move p to 0 with $T_0 = \mathbb{C}^{n-1} \times \mathbb{R}$ and $H_0 = \mathbb{C}^{n-1} \times \{0\}$. Let $z_n = u + iv$ and $v > \varphi(z_1, \ldots, z_{n-1}, u)$.
$(*)$ holds at $0 \Rightarrow \varphi(z_1, \ldots, z_{n-1}, 0)$ has a local minimum at $0 \Rightarrow$ $\text{Hess}_0 \varphi \geq 0$ on $\mathbb{C}^n n - 1 \times \{0\}$. Also, $\text{Hess}_0 \varphi > 0$ on $\mathbb{C}^{n-1} \times \{0\} \Rightarrow \varphi(z_1, \ldots, z_{n-1}, 0)$ has a strict local minimum at 0 $\Rightarrow (*), (**)$ hold at p.
How does the choice of LFT affect Hessians? Let $n = 2$

$$\Phi : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} \frac{D+Ez_1+Fz_2}{A+Bz_1+Cz_2} \\ \frac{G+Hz_1+Iz_2}{A+Bz_1+Cz_2} \end{pmatrix}$$

If we want $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then we need $D = G = 0$. For

$$\det \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} \neq 0$$

we insist that $A = 1$. If

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \approx \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \Phi \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \approx \begin{pmatrix} Ez_1 + Fz_2 \\ Hz_1 + Iz_2 \end{pmatrix}$$

So $\Phi' \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E & F \\ H & I \end{pmatrix}$. Need $\Phi' \begin{pmatrix} 0 \\ 0 \end{pmatrix} : \mathbb{C} \times \mathbb{R} \to \mathbb{C} \times \mathbb{R} \Leftrightarrow H = 0, I > 0$.

$$\begin{pmatrix} 1 & B & C \\ 0 & D & E \\ 0 & 0 & I \end{pmatrix} = \Phi = \Phi_1 \circ \Phi_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & D & E \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} 1 & B & C \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So $\Phi_1 : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} Dz_1 + Ez_2 \\ Iz_2 \end{pmatrix}$. Let $v = \varphi(z_1, u)$ pullback via Φ_1 to $v = \widetilde{\varphi}(z_1, u)$

$$Iv = \varphi(Dz_1 + Ez_2, Iu), \widetilde{\varphi} = \frac{1}{I} \varphi(Dz_1 + E_2 2, Iu), \text{Hess}_0 \widetilde{\varphi} \begin{pmatrix} z_1 \\ u \end{pmatrix} = \frac{1}{I} \text{Hess}_0 \varphi \left(\Phi_1'(0) \begin{pmatrix} z_1 \\ u \end{pmatrix} \right)$$

Do the same pullback with Φ_2.

$$\text{Im} \frac{z_2}{1 + Bz_1 + Cz_2} = \varphi \begin{pmatrix} \frac{z_1}{1+Bz_1+Cz_2} \\ \text{Re} \frac{z_2}{1+Bz_1+Cz_2} \end{pmatrix} = \frac{1}{2} \text{Hess}_0 \varphi \begin{pmatrix} z_1 + \cdots \\ u + \cdots \end{pmatrix} + \cdots = \frac{1}{2} \text{Hess}_0 \varphi \begin{pmatrix} z_1 \\ u \end{pmatrix} + \cdots$$

$$\text{Im} \frac{z_2}{1 + Bz_1 + Cz_2} = \text{Im}(z_2 - Bz_1 z_2 - Cz_2^2 + \cdots) = v - \text{Im}(Bz_1 z_2 + Cz_2^2) + \cdots = v - \text{Im}(Bz_1 u + Cu^2) + \cdots$$

Re-arranging these equations,

$$v = \frac{1}{2}\mathrm{Hess}_0\varphi \mathrm{Im}\begin{pmatrix} z_1 \\ u \end{pmatrix} + \mathrm{Im}(Bz_1z_2 + Cu^2) + \cdots$$

$$\mathrm{Hess}_0\widetilde{\varphi}\begin{pmatrix} z_1 \\ u \end{pmatrix} = \mathrm{Hess}_0\varphi\begin{pmatrix} z_1 \\ u \end{pmatrix} + 2\mathrm{Im}(Bz_1z_2 + Cu^2)$$

$$\mathrm{Hess}_0\widetilde{\varphi}\begin{pmatrix} z_1 \\ 0 \end{pmatrix} = \mathrm{Hess}_0\varphi\begin{pmatrix} z_1 \\ 0 \end{pmatrix}$$

Combine: $\Phi = \Phi_1 \circ \Phi_2$ so that

- $$\mathrm{Hess}_0\widetilde{\varphi}\begin{pmatrix} z_1 \\ 0 \end{pmatrix} = \frac{1}{I}\mathrm{Hess}_0\varphi\left(\Phi'(0)\begin{pmatrix} z_1 \\ u \end{pmatrix}\right)$$
- Remaining terms of $\mathrm{Hess}_0\widetilde{\varphi}\begin{pmatrix} z_1 \\ u \end{pmatrix}$ may be prescribed arbitrarily.

Option 1:

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so the Taylor expansion of $\widetilde{\varphi}$ contains no u terms until at least the 3rd order.
Option 2:

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If $\mathrm{Hess}_0\varphi > 0$ on H_0 get $\mathrm{Hess}_0\widetilde{\varphi} > 0$ on T_0.

Exercise 2.89. Show that this generalizes to higher dimensions.

Let $E \subset \mathbb{R}^n$ be connected, open and bounded with C^2 boundary.

$$E = \{x \mid \rho(x) < 0\}, \rho \text{ is } C^2, d\rho \neq 0 \text{ on } bE$$

Suppose E has another defining function, $\widetilde{\rho}$, $\widetilde{\rho} = \eta\rho$ where $\eta > 0$ on bE.

$$\mathrm{Hess}_p\widetilde{\rho}(x) = \eta(p)\mathrm{Hess}_p\rho(x) + \rho(p)\mathrm{Hess}_p\eta(x) + d_p\rho(x)d_p\eta(x)$$

If $p \in bE$ and x is tangent to bE at p, then $\mathrm{Hess}_p\widetilde{\rho}(x) = \eta(p)\mathrm{Hess}_p\rho(x)$.

Lecture 20. October 28, 2009

When we have been discussing T_a (the real tangent space at a) and H_a (the complex tangent space at a, $H_a \subset T_a$), they are affine spaces, not necessarily vector spaces.
† Assume $E \subset \mathbb{R}^n$ or $\mathbb{C}^n$ is bounded, connected and open with C^2 boundary. E has a C^2 defining function ρ satisfying $d\rho \neq 0$ on bE and $E = \{x \mid \rho(x) < 0\}$.

Theorem 2.90 (1)**.** *In $\mathbb{R}^n$, TFAE*

(1): *$Hess_a\rho \geq 0$ on $T_a^0(bE), \forall a \in bE$*
(2): *E is convex*
(3): *$u : E \to \mathbb{R}$ given by $u(x) = -d(x, bE)$ is convex*
(4): *u is convex near bE*
(5): *$\exists C^\infty \psi : E \to \mathbb{R}$ such that $\psi(x) \to \infty$ as $x \to bE$ and $Hess_a\psi > 0$*
(6): *$E = \cup E_j, E_1 \subset\subset E_2 \subset\subset E_3 \subset\subset \cdots$ all strongly convex (note the notation "$\subset\subset$" means that $\overline{E_i} \subset E_{i+1}$ is compact or E_i is relatively compact in E_{i+1}).*

Definition 2.91. E is **strongly convex** if $\text{Hess}_a\rho > 0$ on $T_a^0(bE), \forall a \in bE$.

Definition 2.92. $\psi : E \to \mathbb{R}$ is **strongly convex** if $\text{Hess}_a\psi > 0, \forall a \in bE$.

Note that this is sometimes referred to as strictly convex.

{domains satisfying †} $\supset^{\text{closed}}$ {convex domains satisfying †} $\supset^{\text{open}}$ {strongly convex domains}

The first set is a Banach manifold.

Theorem 2.93 (2). *In $\mathbb{C}^n$, TFAE*

- **(1'):** *$Hess_a\rho \geq 0$ on $H_a^0(bE), \forall a \in bE$*
- **(2'):** *E is $\mathbb{C}$-convex (and $\mathbb{C}$-linear convexity)*
- **(4'):** *$Hess_a u(w) \geq \frac{||d_a u(w)||^2 + ||d_a(u)(iw)||^2}{u}, \forall a$ near $bE, w \in \mathbb{C}^n$*
- **(6'):** *$E = \cup E_j, E_1 \subset\subset E_2 \subset\subset E_3 \subset\subset \cdots$ all strongly $\mathbb{C}$-convex*

Definition 2.94. E is **strongly $\mathbb{C}$-convex** if $\text{Hess}_a\rho > 0$ on $H_a^0(bE), \forall a \in bE$. *Equivalently,* E is **strongly $\mathbb{C}$-convex** if $\forall a \in bE, \exists$ a LFT Φ such that $\Phi(E)$ strongly convex in a neighborhood of $\Phi(a)$.

f is **convex** if $f(tx + (1-t)y) \geq tf(x) + (1-t)f(y), \forall x, y, 0 \leq t \leq 1$. *Equivalently,* f is **convex** if the epigraph of f is convex. If f is C^2, then f is **convex** $\Leftrightarrow \text{Hess}_a f \geq 0, \forall a$.

Definition 2.95. E is **strongly pseudoconvex** if the strict inequality holds in condition (1") when $w \neq 0$. *Equivalently,* E is **strongly pseudoconvex** if $\forall a \in bE, \exists\Phi$ biholomorphic near a such that $\Phi(E)$ is strongly pseudoconvex in a neighborhood of $\Phi(a)$.

Theorem 2.96 (3). *In $\mathbb{C}^n$, TFAE*

- **(1"):** *$Hess_a\rho(w) + Hess_a\rho(iw) \geq 0, \forall a \in bE, \forall w \in H_a^0(bE)$*
- **(2"):** *E is pseudoconvex*
- **(3"):** *u is locally a clear limit of C^2 functions satisfying ‡*
- **(4"):** *‡ $Hess_a u(w) + Hess_a u(iw) \geq 2\frac{|d_a u(w)|^2 + |d_a u(iw)|^2}{u}, \forall a$ near $bE, w \in \mathbb{C}^n$*
- **(5"):** *$\exists\psi : E \to \mathbb{R}$ C^∞ such that $\psi(z) \to \infty$ as $z \to bE$ and $Hess_a\psi(w) + Hess_a\psi(iw) > 0, \forall a \in E, w \neq 0$*
- **(6"):** *$E = \cup E_j, E_1 \subset\subset E_2 \subset\subset E_3 \subset\subset \cdots$ all strongly pseudoconvex*

These three theorems are all assuming that the boundary is smooth.

Lecture 21.

October 30, 2009 Let $\psi : \mathbb{C}^n \to \mathbb{R}$,

$$\text{Hess}_a\psi(v) = \sum j,k \frac{\partial^2\psi}{\partial z_j \partial \overline{z}_k}(a) v_j \overline{v}_k + \text{Re} \sum_{j,k} \frac{\partial^2\psi}{\partial z_j \partial z_k}(a) v_j v_k$$

$$\text{Hess}_a\psi(v) + \text{Hess}_a\psi(iv) = 2\sum_{j,k} \frac{\partial^2\psi}{\partial z_j \partial \overline{z}_k}(a) v_j \overline{v}_k$$

$$\text{Hess}_a\psi(v) - \text{Hess}_a\psi(iv) = 2\text{Re}\sum_{j,k} \frac{\partial^2\psi}{\partial z_j \partial z_k}(a) v_j v_k$$

These terms are not standardized, but we shall call: $\frac{\partial^2\psi}{\partial z_j \partial \overline{z}_k}(a)$ the $\mathbb{C}-\text{Hess}_a\psi$ and $\frac{\partial^2\psi}{\partial z_j \partial z_k}(a)$ the holomorphic$-\text{Hess}_a\psi$.

$$\text{Hess}\psi = \mathbb{C} - \text{Hess}\psi + \text{Re(holomorphic-Hess}\psi)$$

We can now write the conditions in the previous lecture as:
$(5'')\mathbb{C} - \text{Hess}_a\psi > 0$, "$\psi$ strongly plurisubharmonic"
$(4'')\mathbb{C} - \text{Hess}_a(-\log(-u)) \geq 0$, "$-\log(-u)$ plurisubharmonic"

$(1'')\mathbb{C}-\text{Hess}_a\rho \geq 0$ on H_a^0
Recall: H_a^0 is the $\mathbb{C}$-tangent vector space at a, while H_a is the affine $\mathbb{C}$-tangent spaces at a.
Transformations for Hessians:
Let $M \xrightarrow{\Phi} N \xrightarrow{u} \mathbb{R}$. General rule:

$$\text{Hess}_a(u\circ\Phi)(v) = \text{Hess}_{\Phi(a)}u(\Phi'(a)v) + u'(\Phi(a))\cdot \text{Hess}_a\Phi(v)$$

Usuallly mathematicians prefer when the second term vanishes because the first term is more reminiscent of the standard chain rule. The second term vanishes if:

- Φ is affine (so composing a convex function with an affine function returns a convex function):
  Conditions $(1),(5),(6)$ are directly affine invariant, while $(2),(3),(4)$ are indirectly affine invariant.
  Conditions $(1'),(6')$ are directlly $\mathbb{C}$-affine invariant, while $(4')$ is indirectly affine invariant.
- Φ is holomorphic $\Rightarrow \mathbb{C}-\text{Hess}_a\Phi = 0$
  $$\Rightarrow \mathbb{C}-\text{Hess}_a(u\circ\Phi)(v) = \mathbb{C}-\text{Hess}_{\Phi(a)}u(\Phi'(a)v)$$
  so the composition of a plurisubharmonic function with a holomorphic map is plurisubharmonic. Therefore,
  Conditions $(1''),(5''),(6'')$ are directly holomorphically invariant.
- What if Φ is a LFT? (composition of a convex function with a LFT is not convex)
  For real x, $\Phi(x) = \frac{\text{affine mapping of } f}{A_{0,0}+A_{0,1}x_1+\cdots+A_{0,n}x_n}$.

Lemma 2.97. *$Hess_a\Phi(v) = -2\frac{A_{0,1}v_1+\cdots+A_{0,n}v_n}{A_{0,0}+A_{0,1}a_1+\cdots+A_{0,n}a_n}\Phi'(a)v$.*

Let $a\in M$ and look at the level surfaces through a and $\Phi(a)$. The level surface through a is $(u\circ\Phi)^{-1}(u(\Phi(a)))$.

$$v \text{ is tangent at } a \text{ to } (u\circ\Phi)^{-1}(u(\Phi(a))) \Leftrightarrow (u\circ\Phi)'(a)v = u'(\Phi(a))\Phi'(a)v = 0$$
$$\Leftrightarrow \Phi'(a)v \text{ is tangent at } \Phi(a) \text{ to } u^{-1}\big(u(\Phi(a))\big)$$

So condition $(1),(6)$ are directly LFT-invariant, while the conditions $(2),(3),(4),(5)$ are all indirecly LFT-invariant.
Complex Case:
The same argument works if the vector $v\in H_a^0$. Then
Conditions $(1'),(6')$ are directly LFT-invariant.

Definition 2.98. bE is **strongly $\mathbb{C}$-convex** at a if $\text{Hess}_a\rho > 0$ on H_a^0.

bE is strongly $\mathbb{C}$-convex at $a \Rightarrow \mathbb{C}-\text{Hess}_a\rho > 0$ on H_a^0. $\text{Re}(\text{holo.-Hess}_a\rho) = \text{Re}(\sum\cdots z_jz_k)$ has no particular sign.
We need $\mathbb{C}-\text{Hess}_a\rho(z) + \text{Re}(\text{holo.-Hess}_a\rho(x) > 0, \forall z$.

$$\mathbb{C}-\text{Hess}_a\rho(e^{i\theta}z) + \text{Re}(\text{holo.-Hess})_a\rho(e^{i\theta}) > 0$$

For certain chose of θ, $\mathbb{C}-\text{Hess}_a\rho(e^{i\theta}z) - |\text{holo.-Hess}_a\rho(z)|$.
Need: $|\text{holo.-Hess}_a\rho(z)| < \mathbb{C}-\text{Hess}_a\rho(z), \forall$ non-zero $z\in H_a^0$.

Lecture 22. November 2, 2009

Let $E=\{\rho<0\}$ with a C^2 boundary. Recall: bE is **strongly $\mathbb{C}$-convex** (or $\mathbb{C}$-convex) at $a\in bE \Leftrightarrow \text{Hess}_a\rho > 0$ (or $\text{Hess}_a\rho \geq 0$) on $H_a^0 \Leftrightarrow |\text{holo.-Hess}_a\rho(z) < \mathbb{C}-\text{Hess}_a\rho(z), \forall z\in H_a^0$ non-zero $\Leftrightarrow \frac{|\text{holo.-Hess}_a\rho(z)|}{\mathbb{C}-\text{Hess}_a\rho(z)} < 1, \forall z\in H_a^0$ nonzero $\Rightarrow bE$ strongly pseudoconvex at a.
Replace ρ by $\widetilde{\rho} = \eta\rho$, ($\eta>0$ on bE) $\Rightarrow \text{Hess}_a\widetilde{\rho} = \eta\text{Hess}_a\rho$ on $T_a^0, \mathbb{C}-\text{Hess}_a\widetilde{\rho} = \eta\mathbb{C}\text{Hess}_a\rho$ on H_a^0, and $\text{holo.-Hess}_a\rho = \eta$ $\text{holo.-Hess}_a\rho$ on H_a^0. For $n=2$, $\dim_{\mathbb{C}} H_a^0 = 1$.

(*) Independent of ρ, invariant under rotation, dibdr on z independent of choice of $z \in H_a^0 \setminus \{0\}$. So (*) defines a scalar iinvariant depending on $a \in bE$.
Simple choice of $z = \left(\frac{\partial\rho}{\partial z_2}, -\frac{\partial\rho}{\partial z_1}\right) \in \mathbb{C}^2 \leftrightarrow \frac{1}{2}\left(\frac{\partial\rho}{\partial x_2}, -\frac{\partial\rho}{\partial y_2}, -\frac{\partial\rho}{\partial x_1}, \frac{\partial\rho}{\partial y_1}\right) \in \mathbb{R}^4$. Check that $z, iz \perp \text{grad}\rho$ so $z \in H_a^0$. Need:

$$1 > \frac{\left|(\rho_2 - \rho_1)\begin{pmatrix}\rho_{1,1} & \rho_{1,2}\\ \rho_{2,1} & \rho_{2,2}\end{pmatrix}\begin{pmatrix}\rho_2\\ -\rho_1\end{pmatrix}\right|}{(\rho_2 - \rho_1)\begin{pmatrix}\rho_{1,\overline{1}} & \rho_{1,\overline{2}}\\ \rho_{2,\overline{1}} & \rho_{2,\overline{2}}\end{pmatrix}\begin{pmatrix}\overline{\rho_2}\\ -\overline{\rho_1}\end{pmatrix}} = \frac{\left|-\det\begin{pmatrix}0 & \rho_1 & \rho_2\\ \rho_1 & \rho_{1,1} & \rho_{1,2}\\ \rho_2 & \rho_{2,1} & \rho_{2,2}\end{pmatrix}\right|}{-\det\begin{pmatrix}0 & \overline{\rho_1} & \overline{\rho_2}\\ \rho_1 & \rho_{1,\overline{1}} & \rho_{1,\overline{2}}\\ \rho_2 & \rho_{2,ol1} & \rho_{2,\overline{2}}\end{pmatrix}}$$

Independent of choice of ρ. LFT-invariant.
What happens without absolute values in the norm?

$$M \xrightarrow{\Phi} N \xrightarrow{\rho} \mathbb{R} \text{ and } M \xrightarrow{\hat{\rho}} \mathbb{R}$$

$$\begin{pmatrix}0 & \hat{\rho}_{\overline{k}}\\ \hat{\rho}_j & \hat{\rho}_{j,\overline{k}}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & \Phi_{j,k}\end{pmatrix}\begin{pmatrix}0 & \rho_{\overline{k}}\\ \rho_j & \rho_{j,\overline{k}}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \overline{\Phi}_{k,j}\end{pmatrix}$$

Just need Φ holomorphic. Denominator picks up a factor of $|\det' \Phi|^2$.

$$\det\begin{pmatrix}0 & \hat{\rho}_k\\ \hat{\rho}_j & \hat{\rho}_{j,k}\end{pmatrix} = \det\begin{pmatrix}0 & \sum_l \rho_l \Phi_{l,k}\\ \sum_l \rho_l \Phi_{l,j} & \sum_{l,m} \rho_{l,m}\Phi_{l,j}\Phi_{m,k} - \sum_l \rho_l \frac{A_{0,j}\Phi_{l,k} + A_{0,k}\Phi_{l,j}}{A_{0,0} + A_{1,0}a_1 + \cdots + A_{0,n}a_n}\end{pmatrix}$$
$$= \det\begin{pmatrix}1 & 0\\ 0 & \Phi_{j,k}\end{pmatrix}\det\begin{pmatrix}0 & \rho_j\\ \rho_k & \rho_{j,k}\end{pmatrix}\det\begin{pmatrix}1 & 0\\ 0 & \overline{\Phi}_{k,j}\end{pmatrix}$$

The numerator picks up a factor of $(\det' \Phi)^2$ and the quotient picks up $\frac{\det' \Phi}{\overline{\det' \Phi}}$.

$$\mathcal{B}_{bE} \equiv \frac{\det\begin{pmatrix}0 & \rho_k\\ \rho_j & \rho_{j,k}\end{pmatrix}}{-\det\begin{pmatrix}0 & \rho_{\overline{k}}\\ \rho_j & \rho_{j,\overline{k}}\end{pmatrix}} \frac{dz_1 \wedge dz_2}{\overline{dz_1 \wedge dz_2}}$$

is LFT invariant.
Compare on $\mathbb{C}$ to the Betrami differential $\mu(z)\frac{d\overline{z}}{dz}$. Suppose $f : \mathbb{C} \to \mathbb{C}$ is an orientation-preserving diffeomorphism. Then $\frac{\overline{\partial} f}{\partial f} = \frac{f_{\overline{z}}}{f_z}\frac{d\overline{z}}{dz}$, where $\left|\frac{f_{\overline{z}}}{f_z}\right| < 1$.
Beltrami Equation: Given $|\mu(z)| < 1$, solve $\frac{\overline{\partial} f}{\partial f} = \mu(z)\frac{d\overline{z}}{dz}$.
Special case: Teichmuler differential is $c\frac{\overline{h(z)}}{h(z)}\frac{d\overline{z}}{dz}$, where c is a constant.

Lecture 23. November 4, 2009

Example 2.99. In $\mathbb{C}^2$:

(1) Let $E = \{|z_1|^p + |z_2|^p < 1\}$. Then
$$\mathcal{B}_{bE} = \frac{2-p}{p}\frac{\overline{z_1 z_2}}{z_1 z_2}\frac{dz_1 \wedge dz_2}{\overline{dz_1 \wedge dz_2}} = \frac{2-p}{p}\frac{dz_1 \wedge dz_2}{z_1 z_2}\frac{\overline{z_1 z_2}}{\overline{dz_1 \wedge dz_2}}$$
This is strongly $\mathbb{C}$-convex off the axes when $1 < p < \infty$.

(2) Let $E = \{\text{Im} z_2 > |z_1|^\gamma\}$. Then
$$\mathcal{B}_{bE} = \frac{\gamma-2}{\gamma}\frac{\overline{z_1}}{z_1}\frac{dz_1 \wedge dz_2}{\overline{dz_1 \wedge dz_2}} = \frac{\gamma-2}{\gamma}\frac{dz_1 \wedge dz_2}{z_1}\frac{\overline{z_1}}{\overline{dz_1 \wedge dz_2}}$$
This is strongly $\mathbb{C}$-convex off of $z_1 = 0$ for $\gamma > 1$.

(3) Let $E = \{\mathrm{Im} z_2 > \alpha|z_1|^2 + \mathrm{Re}\beta z_j^2\}$. Then

$$\mathcal{B}_{bE} = \frac{\beta}{\alpha}\frac{dz_1 \wedge dz_2}{\overline{dz_1 \wedge dz_2}}$$

This is strongly convex for $|\beta| < \alpha$. Note that the (2,0) form $dz_1 \wedge dz_2$ has a pole in projective space at ∞.

- $\mathcal{B}_{bE}$ is defined on bE, but in these examples we get an extension of the form: $\frac{\text{rational (2,0) form}}{\text{conjugate of rational (2,0) form}}$.
- In these examples, $|\mathcal{B}_{bE}|$ is constant, but this is not typical.
- bE in $\mathbb{CP}^n$ is not everywhere smooth and strongly $\mathbb{C}$-convex unless in example 1) $p = 2$, 2) $|\gamma| = 2$, 3) $\beta = 0$.

Theorem 2.100 (deTraz-Trepeau/Bolt)**.** *Suppose $\mathcal{B}_{bE} \equiv 0$. Then E is LFT-equivalent to a ball (also local).*

Theorem 2.101 (Bolt)**.** *Suppose $\mathcal{B}_{bE} = k\frac{dz_1 \wedge dz_2}{\overline{dz_1 \wedge dz_2}}$ and $0 < |k| < 1$. Then E is affine-equivalent to example 3 (also local).*

A similar result holds for example 1.
Problems:

(1) When is $|\mathcal{B}_{bE}$ constant?
(2) When is $\mathcal{B}_{bE} = \frac{\text{rational (2,0)}}{conjugate of rational(2,0)}$?

What does $\mathcal{B}$ tell us? Let $a \in E \rightsquigarrow T_o bE = \mathbb{C} \times \mathbb{R}$ and $(z_1, z_2) = (z, u + iv)$. be is given locally by

$$v = f(z,u) = \alpha|z|^2 + \mathrm{Re}\beta z^2 + O(|u|^2) + O(|\beta|^3) + O(|u| \cdot |z|)$$

Recall (from *Lecture 19*), that we can improve this to:

$$v = f(z,u) = \alpha|z|^2 + \mathrm{Re}\beta z^2 + O(|u|^3) + O(|\beta|^3)$$

$\rho(z,w) = f(z,u) - v$. $\mathrm{Hess}_0\rho\begin{pmatrix} z \\ 0 \end{pmatrix} = \alpha|z|^2 + \mathrm{Re}\beta z^2$. If you do not want the defining function in this formula, you can use the second fundamental form to make it independent of the defining function. The new (equivalent) equation is: $\mathbb{I}_0\begin{pmatrix} z \\ 0 \end{pmatrix} = (\alpha|z|^2 + \mathrm{Re}\beta z^2)\left[\frac{d}{dV}\right]$, where $\mathbb{I}_0$ is the second fundamental form.
$\mathcal{B} = \frac{\beta}{\alpha}\frac{dz_1 \wedge dz_2}{\overline{dz_1 \wedge dz_2}}$ (at 0) and $|\mathcal{B}| = \frac{|\beta|}{\alpha}$. $\mathcal{B}$ is strongly $\mathbb{C}$-convex at 0 $\Leftrightarrow a > 0, \frac{|\beta|}{\alpha} < 1$. The levels sets of $\mathbb{I}_0$ are ellipses. The major-to-minor axis ratio ($\frac{1}{\sqrt{1-\mathrm{ecc}^2}}$, where ecc is the eccentricity) is: $\sqrt{\frac{\alpha+|\beta|}{\alpha-|\beta|}} = \sqrt{\frac{1+\frac{|\beta|}{\alpha}}{1-\frac{|\beta|}{\alpha}}} = \sqrt{\frac{1+|\mathcal{B}|}{1-|\mathcal{B}|}}$. The minor axis is given by $\beta z^2 > 0$, i.e. $2\arg(\text{minor axis}) = -\arg\beta$.
Claim: This determines "$\arg\beta$" at 0.
In general, $\mathcal{B}_a = b(a)\frac{dz_1 \wedge dz_2}{\overline{dz_1 \wedge dz_2}}$ determines a map

$$\mathcal{B}_a : \{(x,y) \in T_a^0\mathbb{C}^2 \mid x, y\mathbb{C}\text{-linear ind}\} \to \mathbb{C} \text{ given by } (x,y) \mapsto b(a)\frac{\det(X\vdots Y)}{\det(X \cdots Y)} = \frac{\det^2(X \cdots Y)}{|\det(X\vdots Y)|^2}$$

$\mathcal{B}_a(x,y) = \mathcal{B}_a(y,x) = \mathcal{B}_a(x+y,y)$ and $\mathcal{B}_a(\lambda x, y) = \frac{\lambda}{\bar{\lambda}}\mathcal{B}_a(x,y) = \mathcal{B}_a(x, \lambda y)$.

Back to particular situation: Pick any $y = \begin{pmatrix} \phi \\ t \end{pmatrix} \in T_0 \setminus H_0$ and $x = \begin{pmatrix} z \\ 0 \end{pmatrix} \in H_0$ for $t \in \mathbb{R} \setminus \{0\}$

and $z \neq 0$.

$$\mathcal{B}_0(x,y) = \frac{\beta}{\alpha}\frac{(tz)^2}{|tz|^2} = \frac{\beta z^2}{\alpha |z|^2}$$

We can replace the $\mathcal{B}$'s by $\widetilde{\mathcal{B}}$. Note that $\alpha|z|^2 > 0$. Conclude that $\mathcal{B}_0(x,y) > 0 \Leftrightarrow x \in$ minor axis. This determines "$\arg \mathcal{B}$."

Exercise 2.102. This geometric description of $\mathcal{B}$ works at all $a \in bE$ and is LFT-invariant.

Lecture 24. November 6, 2009

Let S be a $C^2\mathbb{R}$-hypersurface in a 2-dimensional $\mathbb{C}$ manifold M. For $a \in S \subset M$,

$$b(a)\frac{dz_1 \wedge dz_2}{\overline{dz_1 \wedge dz_2}}, |b(a)| < 1 \longleftrightarrow \text{ family of similar ellipses in } H_a^0$$

Two ellipses are similar if they are equivalent via a dilation. In the above equivalence, we can get circles $\Leftrightarrow b(a) = 0$. If $|b(a)| = 1$ we get a family of parallel lines and if $|b(a)| > 1$ then we get a family of hyperbolas.
Suppose $S \subset \mathbb{CP}^2$ is a strongly $\mathbb{C}$-convex hypersurface. $\mathcal{B}_S \longrightarrow$ family of similar ellipses centered at a in H_a, LFT-invariant.
Problem: What "compatibility condition" must $\mathcal{B}_S$ satisfy?
Special Case: Given $\varphi(z_1)\frac{dz_1 \wedge dz_2}{\overline{dz_1 \wedge dz_2}}$, is this $\mathcal{B}_S$ for some strongly $\mathbb{C}$-convex $\mathbb{S} = \{\text{Im}(z_2) = f(z_1)\}$ "rigid hypersurface."

Theorem 2.103. *This happens if and only if*

$$Im\left(\varphi_{\overline{z}\overline{z}} - \overline{\varphi}\varphi_{z\overline{z}} + \frac{\overline{\varphi}\varphi_{\overline{z}}^2 + \varphi\varphi_{\overline{z}}\overline{\varphi}_{\overline{z}} - \overline{\varphi}\varphi_z\varphi_{\overline{z}}}{1 - \varphi\overline{\varphi}}\right) = 0$$

(this is an underdetermined non-linear hyperbolic PDE).

$\mathbb{S} \setminus \{\mathcal{B}_S = 0\}$ is folliated by real curves tangent to the minor axis. Examples?

Returning to $\mathbb{C}^1$:
Metrics on $\mathbb{R}^2 = \mathbb{C}$:
$g = \alpha(z)|dz|^2 + \text{Re}(\beta(z)dz^2)$, with $|\beta| < \alpha$ (written as the hermitian part plus the anti-hermitian part)
$\mathcal{B}_g = \frac{\beta(z)dz^2}{\alpha(z)|dz|^2} = \frac{\beta(z)dz}{\alpha(z)d\overline{z}}$

Definition 2.104. Metrics $g, \widetilde{g}$ are **conformally equivalent** if $\widetilde{g} = \lambda g$, where λ is a positive function of z. Equivalently, $\mathcal{B}_{\widetilde{g}} = \mathcal{B}_g$.

Can we change coordinates so that g is conformally equivalent to the standard metric?
Yes, but it is important that we are in R^2.

$$(\mathbb{C}, g) \xrightarrow{f} (\mathbb{C}, |dw|^2) \text{ has pull-back } f^*(|dw|^2) = (|w_z|^2 + |w_{\overline{z}}|^2)|dz|^2 + \text{Re}(2w_z\overline{w}_z)dz^2$$

$\mathcal{B}_{f^*(|dw|^2)} = \frac{2w_z\overline{w}_z}{|w_z|^2+|w_{\overline{z}}|^2}\frac{dz}{d\overline{z}}$

Definition 2.105. A **conformal dilation** of f, μ_f, is

$$\mu_f = \frac{\overline{\partial} f}{\partial f} = \frac{w_{\overline{z}}d\overline{z}}{w_z dz} = \frac{\overline{\mathcal{B}}_g}{1 + \sqrt{1 - |\mathcal{B}_g|^2}}.$$

Ahlfors-Bers: Assume that $||\mathcal{B}_g||_\infty < 1$ on $\mathbb{C}$. Then $\exists$ an orientation preserving (i.e. the real Jacobian determinant is positive) homeomorphism/difeomorphism $f : \mathbb{C} \to \mathbb{C}$ solving $\frac{\overline{\partial} f}{\partial f} = \frac{\overline{\mathcal{B}}_g}{1+\sqrt{1-|\mathcal{B}_g|^2}}$. f is a diffeomorphism if $\mathcal{B}_g$ is C^1.
Note that $\frac{\overline{\partial} f}{\partial f} = \frac{\overline{\mathcal{B}}_g}{1+\sqrt{1-|\mathcal{B}_g|^2}}$ is called a **Beltrami equation**.

Returning to higher-dimensions: $\mathbb{C}$-dimension ≥ 2
Let $S \subset \mathbb{CP}^n$ a smooth $\mathbb{R}$-hypersurface. $\mathcal{D}_{\mathbb{S}} : \mathbb{S} \to \mathbb{CP}^{n*}$ given by $a \mapsto H_a(\mathbb{S})$. Define $S^* = \mathcal{D}(S)$.

Proposition 2.106. *If $\mathbb{S} = bE$, where E is open and $\mathbb{C}$-convex, then $S^* = b(E^*)$.*

Let's study $\mathcal{D}_S$ (using affinization 3).

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{C}^n \longleftrightarrow \begin{pmatrix} 1 \\ .. \\ z_1 \\ .. \\ \vdots \\ .. \\ z_n \end{pmatrix} \in \mathbb{CP}^n$$

$$(\eta_1, \ldots, \eta_n) \in \mathbb{C}^{n*} \longleftrightarrow (\eta_n : -\eta_1 : \cdots : -\eta_{n-1}, 1) \in \mathbb{CP}^{n*}$$

$$h_\eta = \{z \in \mathbb{C}^n \mid \sum_{j=1}^{n-1} z_j \eta_j = z_n + \eta_n\} = \{z \in \mathbb{C}^n \mid \eta \cdot z = 0\}$$

Why? We like to work near $0 \in S$. $T_0 = \mathbb{C}^{n-1} \times \mathbb{R}, H_0 = \mathbb{C}^{n-1} \times \{0\}$.
Let $\eta = \mathcal{D}_S(z)$. Then $z \in H_z = h_\eta$, i.e. $\sum_{j=1}^{n-1} z_j \eta_j = z_n + \eta_n$.

Lecture 25. November 9, 2009

$S \subset \mathbb{CP}^n$ is a $C^2, \mathbb{R}$-hypersurface with defining function ρ. $\mathcal{D}_S : S \to \mathbb{CP}^{n*}$ is given by $z \mapsto H_z(s)$. $S^{(*)} = \mathcal{D}_S(S) \subset \mathbb{CP}^{n*}$. Let Γ_S be the graph of $\mathcal{D}_S$, so $\Gamma_S \subset \{(z, \eta) \in \mathbb{CP}^n \times \mathbb{CP}^{n*} \mid z \in h_\eta\}$ = the incidence submanifold of $\mathbb{CP}^n \times \mathbb{CP}^{n*}$. Using affinization 3,

$$\{(z, \eta) \in \mathbb{CP}^n \times \mathbb{CP}^{n*} \mid z \in h_\eta\} = \{(z, \eta) \in \mathbb{C}^n \times \mathbb{C}^{n*} \mid \sum_{j=1}^{n-1} z_j \eta_j = z_n + \eta_n\}$$

Affinization 3 excludes vertical hyperplanes so it is fine for local but not global purposes, while affinization 1 is better for global than local purposes.
Focus on $n = 2$.
Work near $0 \in S, T_0 S = \mathbb{C} \times \mathbb{R}, H_0 S = \mathbb{C} \times \{0\}$.

$$\begin{aligned} T_z S &= \left\{ (\zeta_1, \zeta_2) \;\middle|\; 2\mathrm{Re}\left(\frac{\partial \rho}{\partial z_1}(z)(z_1 - \zeta_1) + \frac{\partial \rho}{\partial z_2}(z)(z_2 - \zeta_2) \right) = 0 \right\} \\ H_z S &= \left\{ (\zeta_1, \zeta_2) \;\middle|\; \frac{\partial \rho}{\partial z_1}(z)(z_1 - \zeta_1) + \frac{\partial \rho}{\partial z_2}(z)(z_2 - \zeta_2) = 0 \right\} \\ &= \left\{ (\zeta_1, \zeta_2) \;\middle|\; -\frac{\frac{\partial \rho}{\partial z_1}}{\frac{\partial \rho}{\partial z_2}}(z)\zeta_1 = \zeta_2 - z_2 - \frac{\frac{\partial \rho}{\partial z_1}}{\frac{\partial \rho}{\partial z_2}}(z) z_1 \right\} \\ &= \{(\zeta_1, \zeta_2) \mid \eta_1 \zeta_1 = \zeta_2 + \eta_2\} \end{aligned}$$

$(\eta_1, \eta_2) \in \mathcal{D}_S(z)$. Let $z_2 = u + iv$. Choose

$$\rho = v - f(z, u) = \frac{z_2 - \overline{z_2}}{2i} - \alpha z_1 \overline{z_1} - \frac{\beta}{2} z_1^2 - \frac{\overline{\beta}}{2} \overline{z_1} z + \text{3rd order terms}$$

$\mathcal{D}_S\big(z_1, u+if(z_1,u)\big) = (2i\alpha\overline{z_1}+2i\beta z_1+\cdots, -u+\cdots)$ and $\mathcal{D}'_S(0) : \begin{pmatrix} z_1 \\ u \end{pmatrix} \mapsto \begin{pmatrix} 2i\alpha\overline{z_1}+2i\beta z_1 \\ -u \end{pmatrix}$.

$$\begin{aligned}\mathcal{D}_S \text{ is diffeo. near } 0 &\Leftrightarrow z_1 \mapsto 2-\alpha\overline{z_1}_2 i\beta z_1 \text{ invariant} \\ &\Leftrightarrow |\alpha| \neq |\beta| \\ &\Leftrightarrow \text{strongly } \mathbb{C}-\text{convex or strongly } \mathbb{C}\text{-concave (i.e. } |\beta| > |\alpha|) \\ &\Rightarrow S^{(*)} \text{ is smooth near } 0, \mathcal{D}_S(0) = 0 \in S^{(*)}, T_0S^{(*)} = \mathbb{C}\times\mathbb{R}, \\ &H_0S^{(*)} = \mathbb{C}\times\{0\}, \mathcal{D}_{S^*}\big(\mathcal{D}_S(0)\big) = 0\end{aligned}$$

$\mathcal{D}'_S(0) : H_0S \to H_0S^*$ in the strongly $\mathbb{C}$-convex case (i.e. $|\alpha| > |beta|$), this map is orientation reversing (since $2i\alpha\overline{z_1}$ dominates $2i\beta z_1$) and not $\mathbb{C}$-linear. In general (this in the strongly $\mathbb{C}$-convex case), $\mathcal{D}'_S(z) : H^0_z(S) \to H^0_z(S^{(*)})$ is orientation reversing and not $\mathbb{C}$-linear. $\mathcal{D}_{S^*} \circ \mathcal{D}_S = I$. $\mathcal{D}_S$ is **contact** or quasi- conformal map for sub-Riemannian metrics on S, S^*. All $\mathcal{D}'_S(z)$'s are conjugate linear $\Leftrightarrow \mathcal{B}_S \equiv 0 \Leftrightarrow S$ is LFT-equivalent to part of a sphere.

Remark 2.107. For the sphere, $\beta = 0$ so in some sense the sphere is the most severe strongly $\mathbb{C}$-convex space we can have.

Strongly $\mathbb{C}$-concave case (i.e. $|\beta| > |\alpha|$). Still have $\mathcal{D}_S$ contact, $\mathcal{D}_{S^*} \circ \mathcal{D}_S = I_S$.

Definition 2.108. $\mathcal{D}_S$ is CR if all $\mathcal{D}'_S(z) : H^0_z \to H^0_z S^*$ is $\mathbb{C}$-linear.

$\mathcal{D}_S$ is CR $\Leftrightarrow \mathbb{I}_z$ is anti-hermitian on each $H^0_z S \Leftrightarrow S$ is Levi-flat $\Leftrightarrow^{\text{Frobenius}}$ S is foliated by 1-dimensional $\mathbb{C}$-manifolds.

Remark 2.109. For Levi-flat surfaces, $\alpha = 0$ so in some sense they are the most severe strongly $\mathbb{C}$-concave spaces we can have.

Theorem 2.110. *Suppose $\mathcal{D}_S(U)$ is a C^2 3-dimensional manifold for all relatively open $U \subset S$. Then $\mathcal{D}'_S(z)$ is invertible $\forall z \in S$ (which occurs $\Leftrightarrow S$ is strongly $\mathbb{C}$-convex/concave).*

Proof. Let $V = \{z \in S \mid \mathcal{D}'_S(z) \text{ invertible}\}$ $W = S \setminus \overline{V}$ is relatively open in S. Show that $W = \emptyset$. $\det \mathcal{D}'_S(z) \equiv 0$ on W by definition of W. By Sard's theorem, $\mathcal{D}_S(W)$ has no zero in $S^* \Rightarrow W = \emptyset$. So V is dense in S. $\mathcal{D}_{S^*} \circ \mathcal{D}_S = I$ on $V \Rightarrow \mathcal{D}_{S^*} \circ \mathcal{D}_S = I$ on $S \Rightarrow$ each $\mathcal{D}'_S(z)$ is invertible. □

Lecture 26. November 11, 2009

Brief Look at the Real Case: Let $S \subset \mathbb{RP}^2$ be a smooth curve. $\mathcal{D}_S : S \to \mathbb{RP}^{2*}$ is given by $x \mapsto T_xS$. $\Gamma_S = \{(x,\eta) \in \mathbb{RP}^2 \times \mathbb{RP}^{2*} \mid \eta = \mathcal{D}_S(x)\}$ and $S^{(*)} = \mathcal{D}_S(S)$.
Affine version: $\Gamma_S = \{(x,\eta) \in \mathbb{R}^2 \times \mathbb{R}^{2*} \mid T_xS \text{ is given by } x_1\eta_1 = x_2 + \eta_2\}$
$x \in T_xS \Rightarrow x_1\eta_1 = x_2 + \eta_2$ on Γ_S
η_1 =slope of $T_xS \Rightarrow dx_2 = \eta_1 dx_1$ on Γ_S pulls back to equation on T_xS
$x_1\eta_1 = x_2 + \eta_2 \Rightarrow x_1 d\eta_1 + \eta_1 dx_1 = dx_1 + d\eta_2 \Rightarrow x d\eta_1 = d\eta_2$
So $d\eta_2 = x_1 d\eta_1$ on Γ_S pulls back to hold on $T_\eta S^{(*)}$ when $S^{(*)}$ is smooth.
So the three equations we have are:

(1) $x_1\eta_1 = x_2 + \eta_2$ on Γ_S
(2) $dx_2 = \eta_1 dx_1$ on Γ_S and pulls back to T_xS
(3) $d\eta_2 = x_1 d\eta_1$ on Γ_S and pulls back to $T_\eta S^{(*)}$

The Legendre transform: S =graph of $f \longrightarrow S^*$ =graph of f^* (where f^* is the Legendre transform of f)
Returning to $\mathbb{C}$:
Let $S \subset \mathbb{CP}^2$ be a smooth real hypersurface. $\mathcal{D}_S : S \to \mathbb{CP}^{2*}$ is given by $z \mapsto H_zS$. $\Gamma_S = \{(z,\eta) \in \mathbb{CP}^2 \times \mathbb{CP}^{2*} \mid \eta = \mathcal{D}_S(z)\}$ and $S^{(*)} = \mathcal{D}_S(S)$.
Affine version: $(z,\eta) \in \Gamma_S \Leftrightarrow H_zS$ is given by $z_1\eta_1 = z_2 + \eta_2$.

$z \in H_zS \Rightarrow z_1\eta_1 = z_2 + \eta_2$ on $\Gamma_S \Rightarrow dz_2 = \eta_1 dz_1$
$\widetilde{H}_{(z,\eta)}\Gamma_S \equiv \{(\varphi, \mathcal{D}'_S(\varphi)) \mid \varphi \in H_zS\}$ has $\mathbb{R}$-dimension 2. So $dz_2 = \eta_1 dz_1$ on $\widetilde{H}_{(z,\eta)}\Gamma_S$.
Repeat the argument from the real case: $d\eta_2 = z_1 d\eta_1$ on $\widetilde{H}_{(z,\eta)}\Gamma_S$ and on $H_\eta S^{(*)}$ when this is defined. So the three equations we have are:

(1) $z_1\eta_1 = z_2 + \eta_2$ on Γ_S
(2) $dz_2 = \eta_1 dz_1$ on $\widetilde{H}_{(z,\eta)}\Gamma_S$
(3) $d\eta_2 = z_1 d\eta_2$ on $\widetilde{H}_{(z,\eta)}\Gamma_S$ and on $H_\eta S^{(*)}$ when this is defined

$\mathbb{C}$-Legendre transform: $S = \{v = f(z,u)\} \Rightarrow S^{(*)} = \{v = f^*(z,u)\}$, where f^* is the $\mathbb{C}$-Legendre tranforrm.
Returning to material from the previous lecture:
Let $z_2 = u + iv$ and $S = \{v = \alpha|z_1|^2 + \text{Re}\beta z_1^2 + \cdots\} \Rightarrow \mathcal{D}_S(0) = 0, \mathcal{D}'_S(0) : \begin{pmatrix} z_1 \\ u \end{pmatrix} \mapsto \begin{pmatrix} 2i\alpha\overline{z}_1 + 2i\beta z_1 \\ -u \end{pmatrix}$

$$S^{((*)} \text{ “locally smooth” } \Leftrightarrow |\beta| < |\alpha| \text{ or } |\alpha| < |\beta| \Leftrightarrow \text{ strongly } \mathbb{C}\text{-concave/convex}$$

Get $T_0S^{(*)} = \mathbb{C} \times \mathbb{R}$. Want second order data for $S^{(*)}$ at 0.
$\eta_2 = z_1\eta_1 - z_2 = z_1(2i\alpha\overline{z}_1 + 2i\beta z_1 + \cdots) - u - i\alpha|z_1|^2 - i\text{Re}\beta z_1^2 + \cdots$
Using $z_1 = \frac{-i\alpha\overline{\eta}'_1 - i\overline{\beta}\eta_1}{2(\alpha^2 - |\beta|^2) + \cdots}$:
$\text{Im}\eta_2 = \alpha|z_1|^2 + \text{Re}\beta z_1^2 + \cdots = \cdots = \frac{\alpha|\eta_1|^2 + \text{Re}(\overline{\beta}\eta_1^2}{4(\alpha^2 - |\beta|^2)} + \cdots$
This looks similar to the way S is defined (i.e. $\alpha|z_1|^2 + \text{Re}\beta z_1^2$ is similar to $\frac{\alpha|\eta_1|^2 + \text{Re}(\overline{\beta}\eta_1^2)}{4(\alpha^2 - |\beta|^2)}$): $\mathcal{D}'_S(0)$ maps ellipses in H_0S determined by $\mathbb{I}_0 S$ to elllipses in $H_0S^{(*)}$ determined by $\mathbb{I}_0 S^{(*)}$.
$|\mathcal{B}_S(0)|| = \frac{|\beta|}{\alpha} = |\mathcal{B}_{S^{(*)}}(0)|$. In general, $|\mathcal{B}_{S^{(*)}}(z)| \circ \mathcal{D}_S = |\mathcal{B}_S(z)|$.
Therefore, $S^{(*)}$ is strongly $\mathbb{C}$-convex/concave $\Leftrightarrow S$ is strongly $\mathbb{C}$-convex/concave.
We want to further restrict our choice of projective coordinates:

- we could rotate z_1 so that $\beta \geq 0$ (this gets rid of the issue of $\text{Re}\beta$ versus $\text{Re}\overline{\beta}$)
- we could dilate z_1 (by a real constant) so that $\alpha^2 - |\beta|^2 = \frac{1}{4}$ (this gets rid of the denominator)

Now $S^{(*)} = \{v = \alpha|z_1| + \text{Re}\beta z_1^2 + \text{Re}\gamma z_1^3 + \text{Re}\delta z_1^2\overline{z}_1 + \cdots\}$, note that there are no u^2, z_1u terms. We still have the freedom:

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} \frac{Dz_1 + Ez_2}{1 + Bz_1 + Cz_2} \\ \frac{D^2 z_2}{1 + Bz_1 + Cz_2} \end{pmatrix}, \text{ for } D, C \in \mathbb{R}$$

Theorem 2.111 (Hammond). *Can choose B, E such that $Re\gamma z_1^3, Re\delta z_1^2\overline{z_1}$ match for $S, S^{(*)}$.*

Can't always pin down D:

$$S = \{v = \alpha|z_1|^2 + \text{Re}\beta z_1^2\} \mapsto S \text{ given by } (z, u + iv) \mapsto (D^2 z, Du + iDv) \text{ for } D > 0$$

Lecture 27. November 13, 2009

Let $S \subset \mathbb{CP}^n$ be a $C^2\mathbb{R}$-hypersurface. For $p \in S$, using a LFT we can map $p \mapsto 0$, the $\mathbb{R}$ tangent space to $\mathbb{C}^{n-1} \times \mathbb{R}$ and the $\mathbb{C}$ tangent space to $\mathbb{C}^{n-1} \times \{0\}$. Let $z_n = u + iv$ and $z' = (z_1, \ldots, z_{n-1})$.

$$v = f(z', u) = \sum_{1 \leq j,k \leq n-1} (\alpha_{j,k} z_j \overline{z}_k + \text{Re}\big(sum_{1 \leq j,k \leq n-1} \beta_{j,k} z_j z_k\big) + o(||z'||^2 + u^2)$$

$\mathbb{C}$-convex $\Leftrightarrow |\beta_{j,k} z_j z_k| \leq \sum \alpha_{j,\overline{k}} z_j z_{\overline{k}}$ for all $p \in S$.
strongly $\mathbb{C}$-convex $\Leftrightarrow |\beta_{j,k} z_j z_k| < \sum \alpha_{j,\overline{k}} z_j z_{\overline{k}}$ (for all $p \in S$) when $z' \neq 0 \Rightarrow$ strongly pseudoconvex.

Assume strongly $\mathbb{C}$-convex. Can convert the matrix $(\alpha_{j,k})$ to I. "Diag. of quadratic form"$\Rightarrow$ can convert $(\beta_{j,k})$ to diagonal matrix without changing $(\alpha_{j,k})$. Get $v = \sum_{1\leq j\leq n-1} |z_j|^2 + \operatorname{Re}\sum_{1\leq j\leq n-1} \beta_j z_j^2$. Relax the normalization to $\sum \alpha_j |z_j|^2 + \operatorname{Re}\sum \beta_j z_j^2$ (instead of assuming the $\alpha_i = 1$). Note that we have been assuming that the α_j's are real. Get;

$$\mathcal{D}'_S(0) : \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \\ u \end{pmatrix} \longrightarrow \begin{pmatrix} 2i\alpha_1 \overline{z}_1 + 2i\beta_1 z_1 \\ \vdots \\ 2i\alpha_{n-1}\overline{z}_{n-1} + 2i\beta_{n-1} z_{n-1} \\ -u \end{pmatrix}$$

Note that the first vector is in $\mathbb{C}^{n-1}\times\mathbb{R}$.

$$\begin{aligned} &\text{strong } \mathbb{C}\text{-convexity} \Rightarrow \\ S^{(*)} \text{ is "locally smooth"} &\Leftrightarrow \mathcal{D}_S \text{ is a local diffeomorphism} \\ &\Leftrightarrow |\beta_1| \neq |\alpha_1|, \ldots, |\beta_{n-1}| \neq |\alpha_{n-1}| \end{aligned}$$

This implies $S^{(*)}$ is given by:

$$\operatorname{Im}\beta_n = \sum \frac{\alpha_j |\eta_J|^2}{4(\alpha_j - |\beta_j|^2} + \operatorname{Re}\sum \frac{\overline{\beta_j}\eta_J^2}{4(\alpha_j - |\beta_j|^2} + \cdots$$

Use coordinate rotations to get all $\beta_j \geq 0$. Use coordinate dilations to get $\alpha_j - |\beta_j|^2 = \frac{1}{4}$. Now $S^{(*)}$ satisfies same normalizations as S. Also, $\mathcal{D}'_S$ maps $H_z S$ to $H_\eta S^{(*)}$, but it is not $\mathbb{C}$-linear in the $\mathbb{C}$-convex case (we would need $\alpha_j = 0, \forall j$). After the change in coordinates, $\eta_n = \sum \alpha_j |\eta_j|^2 + \operatorname{Re}\sum \beta_j |\eta_j^2 + \cdots$. $\mathcal{D}'_S$ preserves $\mathbb{I}_z S|_{H_z S}$ up to a multiplicative constant.

Exercise 2.112. Define $\varphi_S : S \to \mathbb{R}$ by $\varphi_S(0) = \prod_{j=1}^{n-1}\left(1 - \frac{|\beta_j|^2}{\alpha_j^2}\right) = \prod_{j=1}^{n-1} \frac{1}{4\alpha_j^2}$ (after the normalization).

(1) General formula (using affine coordinates) for φ_S:

$$\varphi_S(z) = \frac{\det\begin{pmatrix} 0 & 0 & \rho_k & 0 \\ 0 & 0 & 0 & \rho_{\overline{k}} \\ \rho_j & 0 & \rho_{j,k} & \rho_{j,\overline{k}} \\ 0 & \rho_{\overline{j}} & \rho_{\overline{j},k} & \rho_{\overline{j},\overline{k}} \end{pmatrix}}{\det^2\begin{pmatrix} 0 & \rho_{\overline{k}} \\ \rho_j & \rho_{j,\overline{k}} \end{pmatrix}}$$

where the matrix in the numerator is $(2n+2)\times(2n+2)$ and in the denominator is $(n+1)\times(n+1)$, ρ is the defining function and ρ_k is the $\frac{\partial\rho}{\partial z_k}$.

(2) φ_S is LFT-invariant

(3) $n = 2$, $\varphi_S = 1 - |\mathcal{B}_S|^2$

Global Considerations:
$S = bE, E \subset \mathbb{CP}^n$ open and connected. As we have seen previously, S strongly $\mathbb{C}$-convex $\Rightarrow$ $E\mathbb{C}$-convex,$\mathbb{C}$-linearly convex and E^* closed,$\mathbb{C}$-convex,$\mathbb{C}$-linearly convex, $S^{(*)} = b(E^*)$, and we will show today that $S^{(*)}C^2$, is strongly $\mathbb{C}$-convex immersed.

Theorem 2.113. *Let S be given as above. Then $S^{(*)}$ has no self-intersections.*

Lecture 28. November 16, 2009

Lecture 29. November 18, 2009

Bergman Kernel for the unit ball: thinking of the kernel as holomorphic forms and zero,

$$c_n = \frac{dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{w}_1 \wedge \cdots \wedge d\overline{w}_n}{(1 - z\overline{w})^{n+1}}$$

is an (n, n)-form on $B \times B$ invariant under $(z, w) \mapsto (Tz, Tw), T \in \text{Aut}(B)$.
Let $\Omega \subset\subset \mathbb{C}^n$ be an open, connected, strongly $\mathbb{C}$-convex subset and $S = b\Omega$. Let $A(\Omega) = C(\overline{\Omega}) \cap \text{Holo}(\Omega)$ and let μ be a positive cont. multiple of surface measure on S.

Theorem 2.114. *$K \subset \Omega$ compact* $\Rightarrow \max_K |f| \leq C_K ||f||_{L^2(S,\mu)}, \forall f \in A(\Omega)$.

$H(S, \mu) = \overline{A(\Omega)|_S} = L^2(S, \mu)$-closure of $A(\Omega)|_S$

Corollary 2.115. *Each $f \in H(S, \mu)$ has a natural holomorphic extension to Ω.*

Arguing as in the Bergman case, we get a Szegö kernel $k^{Sz.}_{S,\mu}(z, w)$ (abbreviate to $k(z, w)$), which satisfies:

- holomorphic in z with L^2 b.v.
- conjugate holomorphic in w with L^2 b.v.
- $k(w, z) = \overline{k(z, w)}$
- $P^{Sz.}_{S,\mu} : L^2(S, \mu) \xrightarrow[\text{proj.}]{\text{ortho.}} H(S, \mu)$ given by $Pf(z) = \int_S f(w)k(z, w)d\mu(w), z \in \Omega$

There is a problem with the transformation law, which is fixable with a good choice of μ.

Example 2.116. Let Ω =unit ball and μ =Euclidean surface measure. The Szegö kernel turns out to be: $\frac{(n-1)!}{2\pi^n} \frac{(dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{w}_1 \wedge \cdots \wedge d\overline{w}_n)^{\frac{n}{n+1}}}{(1 - z\overline{w})^n}$. This is invariant under Aut$B$. Alternatively, we could work with functions: $T^* f = (f \circ T)(\det T')^{\frac{n}{n+1}}$. When $n = 1$, this agrees with the LFT-transformation law from lecture 2.

Exercise 2.117. A LFT T can be written as $\begin{pmatrix} A_{0,0} & \cdots & A_{0,n} \\ \vdots & & \vdots \\ A_{n,0} & \cdots & A_{n,n} \end{pmatrix}$. Show that

$$\det T' = \frac{-\det(\sim)}{(A_{0,0} + A_{0,1}z_1 + \cdots + A_{0,n}z_n)^{n+1}},$$

where the numerator is usually normalized to 1. Note that the matrix for T is not unique.

We need to be able to interpret $f(z)(dz_1 \wedge \cdots dz_n)^{\frac{n}{n+1}}$. This works out nicely on projective space.

2.2. Line Bundles on $\mathbb{CP}^n$.

Define a **line bundle**, $\mathcal{O}(j, k)$, on $\mathbb{CP}^n$ as follows: Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$. For $E \subset \mathbb{CP}^n = \mathbb{C}^{n+1} \setminus \{0\}/\zeta \sim \lambda\zeta$, let $\widetilde{E} = \pi^{-1}(E) \subset \mathbb{C}^{n+1} \setminus \{0\}$, which is invariant under multiplication by non-zero scalars.

$$\text{section of } \mathcal{O}(j, k) \text{ over } E \leftrightarrow F : \widetilde{E} \to \mathbb{C} \text{ where } F(\lambda\zeta) = \lambda^j \overline{\lambda}^k F(\zeta)$$

Usually $j, k \in \mathbb{Z}$ but it is enough to assume that $j - k \in \mathbb{Z}$ since $\lambda^j \overline{\lambda}^k = |\lambda|^{j+k} e^{i(j-k)\arg\lambda}$,
For holomorphic sections, we would need $k = 0$.
Charts for $\mathbb{CP}^n$: $\mathbb{CP}^n = U_0 \cup \cdots \cup U_n$ where $U_m = \{[\zeta_0 : \cdots : \zeta_n] \in \mathbb{CP}^n \mid \zeta_m \neq 0\}$.
Define $F_m : E \cap U_m \to \mathbb{C}$ by $[\zeta_0 : \cdots : \zeta_n] \mapsto F\left(\frac{\zeta_0}{\zeta_m}, \ldots, 1, \ldots, \frac{\zeta_n}{\zeta_m}\right)$.

Check that $F_l(\zeta) = \left(\frac{\zeta_m}{\zeta_l}\right)^j \overline{\left(\frac{\zeta_m}{\zeta_l}\right)}^k F_m(\zeta)$ is the transition function for $\zeta \in E \cap U_m \cap U_l$.

Important special case: $j = -n-1, k = 0$ so that $F_l(\zeta) = \left(\frac{\zeta_m}{\zeta_l}\right)^{-n-1} F_m(\zeta)$. There is a one to one correspondence:

$$\text{Sections of } \mathcal{O}(-n-1,0) \text{ on } E \leftrightarrow (n,0) - \text{forms on } E$$

Lecture 30. November 20, 2009

$$E \subset \mathbb{CP}^n \leftrightarrow \widetilde{E} \subset \mathbb{C}^{n+1} \setminus \{0\}, \lambda\widetilde{E} = \widetilde{E} \text{ for } \lambda \neq 0$$

$$\Gamma(E,j,k) = \{ \text{ sections of } \mathbb{O}(j,k) \text{ over } E\} \leftrightarrow \{F : \widetilde{E} \to \mathbb{C} \mid F(\lambda\zeta) = \lambda^j \overline{\lambda}^k F(\zeta)\}$$

F could be holomorphic if $k = 0$. F could be positive if $j = k$. Given two sections $F_1 \in \Gamma(E, j_1, k_1), F_2 \in \Gamma(E, j_2, k_2) \Rightarrow F_1F_2 \in \Gamma, j_1 + j_2, k_1 + k_2), \overline{F}_1 \in \Gamma(E, k_1, j_1), F_1\overline{F}_1 \in \Gamma(E, j_1 + k_1, j_1 + k_1), F_1\overline{F}_1 \geq 0$. For $G \in \Gamma(E, j, j), G \geq 0 \Rightarrow G^\alpha \in \Gamma(E, j\alpha, k\alpha)$. For $H \in \Gamma(E, j, k), |H| \equiv \sqrt{H\overline{H}} \in \Gamma\left(E, \frac{j+k}{2}, \frac{j+k}{2}\right)$.

Consider $M \in$SL$(n + 1, \mathbb{C})$ and ψ_M is the LFT associated to M. Let

$$\Gamma(\psi_M E, j, k) \xrightarrow{M^*} \Gamma(E, j, k) \text{ be given by } (M^*F)(\zeta) = F(M\zeta).$$

Recall that ψ_M does not uniquely determine M (i.e. two matrices could both give rise to ψ_M). In particular,

$$\psi_M = \psi_{\widetilde{M}} \Leftrightarrow \widetilde{M} = \omega M, \omega^{n+1} = 1 \Leftrightarrow \widetilde{M}^*F = \omega^{j-k} M^*F$$

The lift to $\Gamma(E, j, k)$ is unique $\Leftrightarrow j - k \in (n + 1)\mathbb{Z}$.
Claim: There is a natural correspondence:

$$\Gamma(E, -n-1, 0) \leftrightarrow (n, 0) - \text{forms over } E$$

How does this work?
On $E \cap U_0$ write $(n, 0)$-form as:

$$f(z_1, \ldots, z_n)dz_1 \wedge \cdots \wedge dz_n$$
$$= f\left(\frac{\zeta_1}{\zeta_0}, \ldots, \frac{\zeta_n}{\zeta_0}\right) d\frac{\zeta_1}{\zeta_0} \wedge \cdots \wedge \frac{\zeta_n}{\zeta_0}$$
$$= \frac{f\left(\frac{\zeta_1}{\zeta_0}, \ldots, \frac{\zeta_n}{\zeta_0}\right)}{\zeta_0^{n+1}} \left(\zeta_0 d\zeta_1 \wedge \cdots \wedge d\zeta_n - \zeta_1 d\zeta_0 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_n + \cdots + (-1)^n \zeta_n d\zeta_0 \wedge \cdots \wedge d\zeta_{n-1}\right)$$

Let $\eta = \zeta_0^{n+1} dz_1 \wedge \cdots \wedge dz_n$.

Exercise 2.118. $dl \wedge \eta = l d\zeta_0 \wedge \cdots \wedge d\zeta_n, \forall l$ linear.

$f(z_1, \ldots, z_n) = F(1, z_1, \ldots, z_n)$ and $F(\zeta_0, \ldots, \zeta_n) \in \Gamma(E \cap U_0, -n-1, 0)$.

Exercise 2.119. This construction give consistent results on each U_m.

Remark 2.120. The bundle associated to $(n, 0)$-forms is called the canonical bundle by algebraic geometers.

Alternate argument: η is $\mathbb{O}(n+1, 0)$-valued $(n, 0)$-form on $\mathbb{CP}^n$. Then $F \in \Gamma(E, -n-1, 0) \Rightarrow F\eta$ is an $(n, 0)$-form.
Given $F \in \Gamma(E, j, k)$ on $E \cap U_0$ write F as $f(z_1, \ldots, z_n)(dz_1 \wedge \cdots \wedge dz_n)^{\frac{-j}{n+1}}(d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n)^{\frac{-k}{n+1}}$.
Note that the multiple-valued problem (of taking fractional powers) is only an issue when we change coordinates. Given $f(z_1, \ldots, z_n) = F(1, z_1, \ldots, z_n)$ and $F(\zeta_0, \ldots, \zeta_n) = \zeta_0^j \overline{\zeta}_0^k f\left(\frac{\zeta_1}{\zeta_0}, \ldots, \frac{\zeta_n}{\zeta_0}\right)$

Example 2.121. Beltrami differential $\mathcal{B}_S = \frac{\beta}{\alpha} \frac{dz_1 \wedge dz_2}{d\overline{z}_1 \wedge d\overline{z}_2} \in \Gamma(S, -3, 3)$ and $|\mathcal{B}_S| \in \Gamma(S, 0, 0)$.

Let $S \subset \mathbb{CP}^n$ be a strongly $\mathbb{C}$-convex hypersurface. Need:

- Norm on $\Gamma(S, -n, 0)$ (since for an invariant norm we need $\Gamma(S, j, k)$ such that $j+k = -n$ and for this to be holomorphic we need $k = 0$)
- $\mathbb{C}$-bilinear pairing between $\Gamma(S, -n, 0)$ and $\Gamma(S^{(*)}, -n, 0)$

For $n = 1$**,** let γ be a curve in the Riemann sphere. The canonical bunddle is $\Gamma(\gamma, -2, 0)$ so $\Gamma(\gamma, -1, 0)$ is the square-root of the canonical bundle. So

$$f(z)\sqrt{dz} \in \Gamma(\gamma, -1, 0), \int_\gamma f(z)\sqrt{dz}\overline{f(z)\sqrt{dz}} = \int |f|^2 |dz|, \text{ and } \int f(z)\sqrt{dz} g(z)\sqrt{dz} = \int fg dz$$

For $n = 2$**,** $F \equiv f(z)(dz_1 \wedge dz_2)^{\frac{2}{3}} \in \Gamma(S, -2, 0)$ so $F\overline{F} \in \Gamma(S, -2, -2)$ and we want $F\overline{F}\mu$ to be a non-negative 3-form. So μ is a positive 3-form on S with values in $\mathbb{O}(2, 2)$. $\int_S F\overline{F}\mu$