Random Walks on Groups

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[1]These are notes from a course that used *Random Walks on Infinite Graphs and Groups* by Wolfgang Woess as its primary text.

[2]These notes were typed during lecture and edited somewhat, so be aware that they are not error free. if you notice typos, feel free to email corrections to swlapan@umich.edu.

Lecture 1. September 9, 2009

Let Ω be a σ-algebra on X and μ a probability measure (so that μ is a measure and $\mu(\Omega) = 1$). Then $\chi : \Omega \longrightarrow U$ is a **random variable**.

Random Walks:

(1) Riemmanian manifold (e.g. sphere, torus, $\mathbb{R}^n$) - if you take a series of random walks (along different geodesics) you want to end up returning to the start, which is recurrence.

(2) Let G be a finitely generated group generated by $S \subset G$ and assume that S is symmetric (i.e. $s \in S \Rightarrow s^{-1} \in S$). A Cayley graph of (G, S) has vertices G with an edge between $\delta, \gamma \in G$ if $\delta = \gamma s$ for some $s \in S$

Sterling's Formula: $k! \sim k^k e^{-k}\sqrt{2\pi k}$ as $k \to \infty$.

Let μ be a probability measure on G and assume that $Supp(\mu) = \{g \in G \mid \mu(g) > 0\}$ generates G. A random walk goes from γ to δ with probability $\mu(\delta\gamma^{-1})$.

Example 0.1. Let $G = \mathbb{Z}$ and $S = \{\pm 1\}$.
L_{2n}, the # of paths of length $2n$, is 2^{2n}.
R_{2n}, the # of path of length $2n$ that come back to the starting point at time $2n$, is $\binom{2n}{n}$

$$\mu_{2n} \equiv \text{ probability return at time } 2n = \frac{R_{2n}}{L_{2n}} = \frac{\text{number returns of length} 2n}{\text{number of paths of length} 2n} = \frac{2n!}{2^{2n} n!^2}$$

Using Sterling's formula, $\mu_n \sim \frac{1}{\sqrt{\pi n}}$ so that $\sum \mu_n = \sum \frac{1}{\sqrt{\pi n}} = \infty$. Hence you return infinitely many times (with probability 1) to your starting point, this is known as **recurrence**.

Example 0.2. Let $G = \mathbb{Z}^2, S = \{\pm(1,0), \pm(0,1)\}$.
$L_{2n} = 4^{2n}$
$R_{2n} = \sum_{k=0}^{n} \frac{(2n)!}{k!^2 (n-k)!^2}$
Since k-times to the right and $n-k$-times up $\Rightarrow$ k-times to the left and $n-k$-times down.

$$\begin{aligned} \mu_{2n} &= \frac{1}{4^{2n}} \sum_{k=0}^{n} \frac{(2n)!}{(k!(n-k)!)^2} \\ &=^* \frac{1}{4^{2n}} \frac{(2n)!}{n!^2} \binom{2n}{n} \\ &\sim \frac{1}{\pi n} \text{by Sterling's formula} \end{aligned}$$

*Note: $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$

Example 0.3. Let $G = \mathbb{Z}^3, S = \{\pm e_i\}$, where e_i is 1 in the ith place.
$L_{2n} = 6^{2n}$
$R_{2n} = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{(2n)!}{(k!j!(n-j-k)!)^2}$

$$\begin{aligned} \mu_{2n} &= \frac{1}{6^{2n}} \sum_{0 \le k, j \& k+j \le n} \frac{(2n)!}{k!^2 j!^2 (n-k-j)!^2} \\ &\sim \frac{\sqrt{2}}{n^1.5} (\frac{2\pi}{3})^{\frac{1}{3}}. \end{aligned}$$

So that $\sum \mu_n < \infty \Rightarrow$ this is not recurrent.

Theorem 0.4 (Polya)**.** *A simple random walk on $\mathbb{Z}^d$ is recurrent if and only if $d \le 2$.*

Theorem 0.5 (Varoponlos)**.** *Let G be finitely generated. A symmetric random walk by μ (where the support of μ generates G) is recurrent if and only if G is finite, $G \supset \mathbb{Z}$ or $G \supset \mathbb{Z}^2$ as a subgroup of finite index.*

Definition 0.6. A probability measure is **symmetric** if $\mu(g) = \mu(g^{-1})$.

Theme 1: Properties of random walks and how they relate to properties of G

Lecture 2. September 11, 2009

Harmonic functions satisfy the averaging property:

Definition 0.7. Let $f : G \longrightarrow \mathbb{R}$ and $Pf(x) = \sum_{\gamma \in G} \mu(\gamma^{-1}x)f(x)$. Then f is **harmonic** if $f = Pf$ and f is **superharmonic** if $Pf \leq f$.

Each term in the sum for Pf is a point times the likelihood that point is reached.

Theorem 0.8. *(G, μ) is recurrent if and only if every superharmonic function is constant.*

Conversely, there are lots of harmonic functions. Let f is harmonic on D^2:

$$f(x) = \int_{S^1} P(x, \zeta)f(\zeta)d\lambda(\zeta), \text{ where } d\lambda(\zeta) \text{ is the Lebesgue measure}$$

$$P(x, \zeta) = \frac{1 - |x|^2}{|x - \zeta|}\omega, \text{ where } \omega \text{ is a constant and } P(x, \zeta) \text{ is the Poisson kernel,}$$

Theme 2: Boundaries of groups and the Poisson representation of harmonic functions If we start a random walk, we expect that it will go towards the boundary.

Example 0.9. Let $G = \pi_1($ compact surface $)$. Then the Poisson boundary is a circle.

Example 0.10. Let $\Gamma \subset Isom(\mathbb{H}^2) = PSL(2, \mathbb{R})$. Then

$$\partial\Gamma = \partial(PSL(2, \mathbb{R}) = S^1 = PSL(2, \mathbb{R})/(\text{upper triangular matrices.})$$

If G/Γ is compact and G is locally compact, then Γ is discrete subgroup of G.

Definition 0.11. Γ is a **cocompact lattice** in G if G/Γ is compact.

Definition 0.12. $\Gamma \subset G$ is a **lattice** in G if $vol(G/\Gamma) < \infty$, where the volume is given by the Haar measure.

Example 0.13. Suppose $\Gamma \subset G_1, G_2$ is a lattice of both G_1 and G_2. Then $\partial G_1 = \partial G_2$ and so G_1 and G_2 must be close (Mostow-Marquilis rigidity theorem).

LAW OF LARGE NUMBERS:

Classical probability: Let $\chi : \Omega \longrightarrow \mathbb{R}$ (function for random variables) and μ be a probability measure. Given a sequence of random variables $x_1, x_2, \ldots, x_n, \ldots$, assume that the x_i are **independent**

$$\mu\{\omega \mid x_i(\omega) < a \text{ and } x_j(\omega) < a\} = \mu\{\omega \mid x_i(\omega) < a\} \cdot \mu\{\omega \mid x_j(\omega) < a\}$$

and **identically distributed**

$$(x_i)_*(\mu) = (x_j)_*(\mu)$$

Then $x_i(\mu)(A) = \mu(x_i^{-1}(A))$.

Theorem 0.14. *Suppose $\alpha \equiv \int_\Omega x_i d\mu < \infty$ (finite first moments). Then*

$$\frac{1}{n}(x_1 + x_2 + \cdots + x_n) \to \alpha = \int_\Omega x_i d\mu = \int_\Omega x_j d\mu \textit{ is the } \textbf{\textit{expectation}} \textit{ of } x_i$$

Theorem 0.15 (Ergodic theorem)**.** *Let Z be a measure space, μ a probability measure, and $T : Z \longrightarrow Z$ such that $T_*\mu = \mu$. Let f be a measurable function on Z. Then*

$$\frac{1}{n}\big(f(x) + f(Tx) + \cdots + f(T^n x)\big) \xrightarrow[ptwise]{a.e.} f^+ = \begin{cases} constant, & \text{if } T \text{ is Ergodic} \\ \int_Z f d\mu = E(f), & \text{otherwise} \end{cases}$$

Definition 0.16. T is **Ergodic** if the only T-invariant measurable sets in X have measure equal to 0 or 1.

Theorem 0.17. $\lim_{n\to\infty} \frac{1}{n} \log ||x_1 x_2 \cdots x_n|| \to \lambda =$ ***Lyapunov exponent***

Remark 0.18. The multiplicative Ergodic theorem is a similar to the Ergodic theorem. It provides the theoretical background for computation of Lyapunov exponents of a nonlinear dynamical system.

Theme 3: Random Groups.
Let S be a finite set and let m be the number of elements in S.
Let Γ have an S-finite set of generators with finite presentation.
Assume that all relations have length 3. For instance, if $S = \{a_1, \ldots, a_n\}$, then all relations can be written with 3 elements as in the equation $a_1 a_7 a_3 = 1$.
Let d be the density and let $(2n-1)^{3d}$ be approximate the number of relations.

Theorem 0.19. *Let $d < \frac{1}{2}$ and $\mathcal{P}(n,d)$ be the set of groups describable by the set-up above.*

$$\lim_{m\to\infty} \frac{\#\{\Gamma \in \mathcal{P}(n,d), \Gamma \textit{ is an infinite Gromov hyperbolic group}\}}{\#\mathcal{P}(n,d)} = 1$$

If $\frac{1}{3} < d < \frac{1}{2}$, then each Γ is a Kazdan group.

Lecture 3. September 14, 2009

Definition 0.20. Let X be a finite or countable set, called the **state space**. Let $P = \big(p(x,y)\big)_{x,y\in X}$, where $0 \le p(x,y) \le 1$ is the probability to go from x to y and $\sum_{y\in X} p(x,y) = 1$. This is a **Markov chain**.

Definition 0.21 (Alternative)**.** A **Markov chain** is a sequence of random variables $x_1, x_2, x_3, \ldots$ with the **Markov property** that given the present state, the future and past states are independent. Formally,

$$\Pr(x_{n+1} = y \mid x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n) = \Pr(x_{n+1} = y | x_n = y_n).$$

The possible values of x_i form a countable set X called the **state space** of the chain.

Remark 0.22. Markov chains are often described by a directed graph, where the edges are labeled by the probabilities of going from one state to the other states.

Let $\Omega = X^{\mathbb{N}}$ and $Z_n : \Omega \longrightarrow X$ be the projection of Ω to its nth factor. Let $x \in X$ be the "starting point." Endow Ω with the following probability measure:

$$\mathbb{P}_x[z_0 = x_0, z_1 = x_1, \ldots, z_n = x_n] = \delta_x(x_0) p(x_0, x_1) p(x_1, x_2) \cdots p(x_{n-1}, x_n)$$

Let E_x be the the expectation of $\mathbb{P}_x$ and $f : \Omega \longrightarrow \mathbb{R}$ a measurable function. Then $E_x(f) = \int_\Omega f d\mathbb{P}_x$ (X, P) (or $\{Z_n\}$) is a Markov Chain. Define $p^{(n)}(x,y)$ as the probability of going from state x to state y in n steps, $\forall n \in \mathbb{N}$. Formally,

$$p^{(n)}(x,y) = \mathbb{P}_x[z_n = y \mid z_0 = x]$$

Notice that,

$$p^{(1)}(x,y) = p(x,y), p^{(2)}(x,y) = \sum_{z\in X} p(x,z)p(z,y), p^{(n)}(x,y) = \sum_{z\in X} p^{(k)}(x,z)p^{(n-k)}(z,y), \forall 0 < k < n$$

Definition 0.23. (X,P) is **irreducible** if $\forall x, y \in X, \exists n$ such that $p^{(n)}(x,y) > 0$.

Remark 0.24. From now on we shall assume that all Markov chains are irreducible.

Definition 0.25. Let X be a state space, $x, y \in X$, and $z \in \mathbb{C}$.

$$G(x,y|z) \equiv \sum_{n=0}^{\infty} p^{(n)}(x,y) \cdot z^n \text{ and } G(x,y) \equiv G(x,y|1)$$

$G(x,y|z)$ is **Green's function**.

Green's functions are functions which, in some sense, generate harmonic functions.

Lemma 0.26. *Let $z \in \mathbb{R}_{>0}$. Then $\forall x_1, x_2, y_1, y_2 \in X$,*

$$G(x_1, y_1|z) \text{ converges } \Leftrightarrow G(x_2, y_2|z) \text{ converges.}$$

Proof. By irreducibility of (X, P), $\exists k, l \in \mathbb{N}$ such that $p^{(k)}(x_1, x_2) > 0$ and $p^{(l)}(y_2, y_1) > 0$. Hence for $z > 0$,

$$\begin{aligned} G(x_1, y_1|z) &= \sum_{n=0}^{\infty} p^{(n)}(x_1, y_1) \cdot z^n \\ &\geq \sum_{n=0}^{\infty} p^{(k)}(x_1, x_2) p^{(n)}(x_2, y_2) p^{(l)}(y_2, y_1) \cdot z^{n+k+l} \\ &= p^{(k)}(x_1, x_2) p^{(l)}(y_1, y_2) G(x_2, y_2|z) \cdot z^{k+l} \\ G(x_2, y_2|z) &= \sum_{n=0}^{\infty} p^{(n)}(x_2, y_2) \cdot z^n \\ &\geq p^{(k)}(x_1, x_2) p^{(l)}(y_2, y_1) \sum_{n=0}^{\infty} p^{(n)}(x_2, y_2) \cdot z^{n+k+1} \\ &\geq \sum_{n=0}^{\infty} p^{(n+k+l)}(x_1, x_2) \cdot z^{n+k+l} \end{aligned}$$

The last term is all except for a finite set of the elements in the sum that gives $G(x_1, y_1)$, so if that is bounded, so is $G(x_1, y_1|z)$. Hence if one of these is finite, so is the other. □

Corollary 0.27. *All $G(x,y|z)$ have the same radius of convergence, which is given by $\tau(P) \equiv \frac{1}{\rho(P)}$, where $\rho(P) = \limsup p^{(n)}(x,y)^{\frac{1}{n}} \in [0,1]$ is the spectral radius of (X, P).*

Remark 0.28. $\rho(P) > 0$ and $p^{(r)}(x,x) > 0$ for some r. Therefore

$$p^{(nr)}(x,x) \geq p^{(r)}(x,y)^n \text{ and } \rho(P) \geq p^{(r)}(x,x)^{\frac{1}{r}}$$

Definition 0.29. The **period** of P, denoted $d(P)$, is $\gcd\{n \mid p^{(n)}(x,x) > 0\}$. The period is independent of x.

Example 0.30. For a simple random walk on $\mathbb{Z}$, $d = 2$.

Definition 0.31. P is **aperiodic** if $d(P) = 1$.

Fix $o \in X$ and set $Y_j = \{x \in X \mid p^{(nd+j)}(o,x) > 0 \text{ for some } n \geq 0\}$. Then $o \in Y_0$.

Lemma 0.32. *$p^{(n)}(x,x) \leq \rho(P)^n$ and $\lim_{n\to\infty} p^{(nd)}(x,x)^{\frac{1}{nd}} = \rho(P)$.*

Proof. Let $a_n = p^{(nd)}(x,x)$, where $0 \leq a_n \leq 1$.

$$p^{(nd)}(x,x) p^{(md)}(x,x) \leq \sum_{y \in X} p^{(nd)}(x,y) p^{(md)}(y,x) = p^{(m+n)d}(x,x)$$

Therefore the a_1 satisfy the crucial "sub additive/multiplicative" property:

$$a_m \cdot a_n \leq a_{m+n}$$

(1) Claim: $\exists n_0$ such that $a_n > 0, \forall n \geq n_0$.
Let $N_x = \{n \mid a_n > 0\}$. Then if $a, b \in N_x \Rightarrow a+b \in N_x \Rightarrow 1 = \gcd N_x = n_1 - n_2$, where $n_1, n_2 \in N_x$. Indeed, can write $\gcd N_x = a \cdot n_1 * - b \cdot n_2 *$ for some $n_1 *, , n_2 * \in N_x$.
If $n_2 = 0$, then $n_1 = 1$ ($N_x = \mathbb{N}$). If $n_2 \neq 0$, then set $n_0 = n_2^2$. If $n \geq n_0$, write $n = qn_2 + r = (q-r)n_2 + rn_1 = qn_2 + r(n_1 - n_2) = qn_2$, where $q \geq n_2 > r$ and since $n_1 - n_2 = 1$.

(2) Fix $m \in N_x$. Let $n \geq n_0 + m$. Write $n - n_0 = q_n m + r_n *$, with $0 \leq r_n * < m$. Then $n = q_n m + (n_0 + r_n *)$. Let $r_n \equiv n_0 + r_n *$. Then $n_0 \leq r_n < n_0 + m$. Set $b = b(m) = \min\{a_r \mid n_0 \leq r < n_0 + m\}$. Note: $a_n \geq a_m^{q_n} a_{r_n}$. Then $a_n^{\frac{1}{n}} \geq a_m^{\frac{q_n}{n}} a_{r_n}^{\frac{1}{n}}$. As $n \to \infty, \frac{q_n}{n} \to \left(\frac{1}{m}\right)^m \Rightarrow a_m^{\frac{1}{m}} \leq \liminf a_n^{\frac{1}{n}} \leq \rho(P)^d$. Let $m \to \infty$. Then $\limsup a_m^{\frac{1}{m}} \leq \rho(P)^{\frac{1}{d}} \leq \liminf a_n^{\frac{1}{n}}$. Thus $a_m^{\frac{1}{m}}$ converges.

□

Lecture 4. September 16, 2009

Example 0.33. A simple random walk on $\mathbb{Z}$ with generators ± 1 has $d = 2$ and

$$\rho^{(2n)} = \mu_{2n} \sim \frac{1}{\sqrt{\pi n}} \Rightarrow p^{(2n)}(x,x)^{\frac{1}{2n}} \sim \left(\frac{1}{\sqrt{\pi n}}\right)^{\frac{1}{2n}} \to 1 \text{ as } n \to \infty$$

Markov chain: Let $X = \mathbb{Z}$, then $p(x,y) = \begin{cases} \frac{1}{\text{elements in gen. set}} & \text{if } xy^{-1} \in \text{Generators} \\ 0 & \text{else} \end{cases}$.

More generally, suppose X is a graph.

Definition 0.34. A **simple random walk** on X is the Markov chain with:

$$\text{state space } \{\text{vertices of } X\} \text{ and } p(x,y) = \begin{cases} \frac{1}{\deg x} & \text{if } y \text{ adjacent to } x \\ 0 & \text{else} \end{cases}$$

Definition 0.35. (X, P) is **recurrent** if the Green's function $G(x,y) = \infty, \forall x, y \in X$. Otherwise (X, P) is **transient**.

If $\rho(P) < 1 \Rightarrow G(x, y|1)$ converges $\Rightarrow (X, P)$ is transient.

Corollary 0.36. *If $\rho(P) < 1 \Rightarrow (X, P)$ is transient.*

The converse fails, for instance a simple random walk on $\mathbb{Z}^2$ is a counter-example.
Let (X, P) be a Markov chain. Given $f : X \longrightarrow \mathbb{R}$, define

$$P(f)(x) = \sum_y p(x,y) f(y), \text{ where we assume } P(f)(x) \text{ is finite.}$$

Definition 0.37. f is **harmonic** if $Pf = f$ and **superharmonic** if $Pf \leq f$.

Example 0.38. Constant functions are harmonic and $G_y(x) = G(x, y)$ is superharmonic for any fixed $y \in X$.

Proof. That $G_y(x)$ is superharmonic

$$\begin{aligned} G_y(x) &= \sum_{n=0}^{\infty} p^{(n)}(x,y) \\ P(G_y)(x) &= \sum_{x} p(x,z)G_y(z) \qquad\qquad = \sum_{z} p(x,z)\Big(\sum_{n=0}^{\infty} p^{(n)}(z,y)\Big) \\ &= \sum_{n=0}^{\infty}\sum_{z} p(x,y)p^{(n)}(z,y) \qquad \le \sum_{n=0}^{\infty} p^{(n+1)}(x,y) \\ &\le G_y(x) \end{aligned}$$

□

Lemma 0.39. *Let $P^n = P \circ \cdots \circ P$ (n-times). Then $P^n(f)(x) = \sum_y p^{(n)}(x,y)f(y)$.*

Proof. Use induction on n. For $n = 1$ this is by definition. Assume this is true for n.

$$\begin{aligned} P(P^n(f))(x) &= \sum_z p(x,z)P^n(f)(z) \\ &= \sum_z \sum_t p(x,z)p^{(n)}(z,t)f(t) \\ &= \sum_t p^{(n+1)}(x,t)f(t) \end{aligned}$$

□

Minimum Principle:
Suppose f is superharmonic and $\exists x_0 \in X$ such that $f(x_0) \le f(x), \forall x \in X \Rightarrow f$ is constant.

Proof. Note that $f \ge g \Rightarrow Pf \ge Pg$. So, $f \ge Pf \ge P^2 f \ge \cdots$. Therefore, $f(x_0) \ge \sum_y p^{(n)}(x_0,y)f(y), \forall n \in \mathbb{N} \Rightarrow \exists y$ such that $f(y) < f(x_0)$. Contradiction. □

Theorem 0.40. *(X,P) is recurrent $\Leftrightarrow$ all non-negative superharmonic functions are constant.*

Proof. Error in proof: wrote $\alpha > \beta\alpha \Rightarrow \beta = 1$ in proof that Markov process transient implies there exists a superharmonic $h \ge 0$ non-constant.
Claim: ($\Leftarrow$) Assume (X,P) is transient. Then $\infty > G_y(x) \ge 0$ is superharmonic. $G_y(x)$ is not harmonic since:

$$\begin{aligned} PG_y(x) &= \sum_z p(x,z)G_y(z) = \sum_z \sum_{n=0}^{\infty} p(x,z)p^{(n)}(z,y) \\ &= \sum_{n=0}^{\infty} p^{(n+1)}(x,y) = G(x,y) - p^{(0)}(x,y) \le G(x,y) \end{aligned}$$

Now we need to show that $G_x(y)$ is non-constant. Suppose $p^{(k)}(x,x) > 0$ and $p^{(l)}(y,x) > 0$ for some k,l. Then $G(x,x) \ge p^{(k)}(x,x)p^{(l)}(y,x)G(x,y)$, since
$p^{(n)}(x,x) \ge p^{(k)}(x,x)p^{(n-k-l)}(x,y)p^{(l)}(y,x)$. If $G(x,x) = G(x,y)$, then $p^{(k)}(x,x) = p^{(l)}(x,y) = 1$ and $p^{(2k)}(x,x) = 1 \Rightarrow G(x,x) = \infty \Rightarrow (X,P)$ is recurrent. Contradiction.
($\Rightarrow$) Suppose that (X,P) is recurrent and $f \ge Pf$ and f is non-constant. Set $g = f - Pf$. Claim: $P = c$ (i.e. f is harmonic). Suppose $g(y) > 0$ for some $y \in X$. Let $x \in X$. Then

$$\sum_{k=1}^{n} p^{(k)}(x,y)g(y) \le \sum_{k=0}^{n} P^k(g)(x) = -P^{n+1}(f)(x) + f(x) \le f(x).$$

Hence $G(x,y) \leq \frac{f(x)}{g(y)}$. Since we can take $\lim_{n\to\infty} -P^{n+1}(f)(x) + f(x)$. So $G(x,y)$ is finite, but we assumed recurrent. Contradiction. $\square$

Stopping Time:
$S^y(x) \equiv \min\{n \geq 0 | Z_n = y\}$, where $Z_0 = x$. Set

$f^{(n)}(x,y) = \mathbb{P}_x[s^y = n], F(x,y|z) = \sum -n = 0^\infty f^{(n)}(x,y)z^n$ and $F_y(x) = F(x,y) = F(x,y|1)$

$F_y(x)$is the probability of ever reaching y starting from x. Note that $F(x,x|z) = 1$. Set $t^x = \min\{n \geq 1 \mid z_n = x\}$. Note: similar to S^y but not the same since $n = 0$ is not allowed. Set

$U(x,x|z) = \sum_{n=0}^{\infty} \mathbb{P}_x[t^x = n]z^n, U(x,x) = U(x,x|1) =$ the probability of ever returning x from x

Properties:

(1) $G(x,x|z) = \frac{1}{1-U(x,x|z)} \in [0,\infty]$
(2) $G(x,y|z) = F(x,y|z)G(y,y|z)$
(3) $U(x,x|z) = \sum_y p(x,y)zF(y,x|z)$
(4) If $y \neq x$: $F(x,y|z) = \sum_w p(x,,w)zF(w,y|z)$

Lecture 5. September 18, 2009

P is recurrent, then every superharmonic function is constant. We proved last time

Lemma 0.41. *Every superharmonic function is harmonic. f is superharmonic implies $G(x,y) \leq \frac{f(x)}{g(y)}$, where $g = f - Pf$ and y such that $g(y) > 0$.*

Finish proof: Step 2 -
Claim: Every harmonic function is constant.
Let f be harmonic and $M = f(x)$ for some $x \in X$. Set $h(y) = min\{M, f(y)\}$.

Lemma 0.42. *h is superharmonic*

h is harmonic $\Rightarrow -h$ is harmonic. $-h$ has a max/min so $\pm h$ has a positive minimum, therefore h is constant.

Proof of lemma:

$$(Ph)(x) = \sum_y p(x,y)h(y) \leq \sum_y p(x,y)M = M$$
$$(Ph)(x) = \sum_y p(x,y)h(y) \leq \sum_y p(x,y)f(y) = f(y)$$

Hence, $Ph \leq h$.

Stopping Time
See handout.

Proof Of Properties. Property 1: If $n \geq 1$

$$p^{(n)}(x,x) = \sum_{k=0}^{n} \mathbb{P}_x[t^x = k]p^{(n-k)}(x,x),\ p^{(0)}(x,x) = 1,\ \text{and } \mathbb{P}_x[t^x = 0] = 0$$

Equation 1 is equivalent to: $G(x,x|z) = G(x,x|z)U(x,x|z) + 1$.

$$p^{(n)}(x,x)z^n = \sum_k \mathbb{P}[t^x = k]z^k p^{(n-k)}(x,x)z^{n-k}$$

Property 2-4:
Similar proofs $\square$

Corollary 0.43. *$F_y(x) \equiv F(x,y)$ is superharmonic.*

Proof. Use property 4: Assume $x \neq y$

$$\begin{aligned} P(F_y)(x) &= \sum_z p(x,z)F_y(z) \\ &= \sum_z p(x,z)F(z,y) \\ &= F_y(x) \end{aligned}$$

If $x = y$, then $F_y(x) = F(x,x) = 1$, so

$$1 = \sum_z p(y,z) \geq \sum_x p(y,z)F_y(z) = (PF_y)(y) = F_y(y)$$

□

Corollary 0.44. *(X,P) is recurrent if and only if $U(x,x) = 1$.*

Proof. Use equation 1. If $U(x,x) = 1$, then $G(x,x) = \infty$. □

Proposition 0.45.

(1) *If (X,P) is recurrent, then $F(x,y) = 1$ and $\mathbb{P}_x\big[z_n = y \text{ for } \infty - \text{many } n\big] = 1$*
(2) *If (X,P) is transient, then $\forall$ finite $A \subset X, \forall x \in X$,*

$$\mathbb{P}_x\big[z_n \in A \text{ for infinitely many } n\big] = 0.$$

Proof.

(1) (X,P) is recurrent and F_y is superharmonic, so F_y is constant (all non-negative superharmonic functions are constant). Hence $F_y(x) = F_y(y) = 1$. Set $V(x,y) = \mathbb{P}_x\big[z_n = y \text{ for infinitely many } m\big]$. $V(x,y) = F(x,y)V(y,y) \leq V(y,y)$ (this is intuitively clear). Set $V_m(x,x) = \mathbb{P}_x\big[z_n \text{ visits } x \text{ at least } m\text{-times}\big]$. Then $V_1(x,x) = 1, V_m(x,x) = U(x,x)V_{m-1}(x,x)$.

$$V(x,x) = \lim_{m\to\infty} V_m(x,x) = \lim_{m\to\infty} U^{m-1}(x,x)V_1(x,x) = \lim_{m\to\infty} U^{m-1}(x,x) = \begin{cases} 1 & \text{if } U(x,x) = 1 \\ 0 & \text{else} \end{cases}$$

(2) Let $A \subset X$ finite and recall $V(x,x) = \begin{cases} 1 & \text{if recurrent} \\ 0 & \text{if transient} \end{cases}$. Then:

$$\mathbb{P}_x[z_n \in A \text{ infinitely many times}] \leq \sum_{y\in A} V(x,y) \leq \sum_{y \in A} V(y,y) = 0 \text{ if transient.}$$

□

Lecture 6. September 21, 2009

Corollary 0.46. *If X is finite, then it is recurrent.*

Example 0.47. Tree T_m with $x \in T_m, \deg(x) = m$ (i.e. m edges at every vertex). For $x, y \in T_m$, $\pi(x,y)$-"geodesic" from x to y (where the geodesic is the connection by edges of the shortest length).

Lemma 0.48. *$w \in \pi(x,y)$, i.e. w is a vertex on the geodesic from x to y. $F(x,y|z) = F(x,w|z)F(w,y|z)$, i.e. a random walk from x to y has to pass through w. $f^{(n)}(x,y) = \sum_{k=0}^n f^{(k)}(x,w)f^{(n-k)}(w,y)$. This implies the claim.*

Proposition 0.49. *Simple random walk on T_m has*

$$G(x,y|z) = \frac{2(M-1)}{M-2+\sqrt{M^2-4(M-1)z^2}}\left(\frac{M-\sqrt{M^2-4(M-1)z^2}}{2(M-1)z}\right)^{d(x,y)}$$

$$\rho(P) = \frac{2\sqrt{M-1}}{M} \text{ and } z > \frac{M}{2\sqrt{M-1}} \Rightarrow \text{ is not in } \mathbb{R}$$

Proof. $F(x,y|z)$ is independent of x and y as long as they are adjacent (by symmetry of the homogeneous tree). So $F(x,y|z) = F(z)$. Let $w \sim x$ denote that w is adjacent to x.

$$\begin{aligned}
F(z) &= F(x,y|z) \\
&= \sum_w p(x,w)zF(w,y|z)(\text{ by formula 4 on the Stopping Time handout.}) \\
&= \sum_{w\sim x} \frac{1}{M} zF(w,y|z) \\
&= \sum_{w\sim x} \frac{1}{M} F(z)^{d(y,x)} \text{ *see below} \\
&= \frac{z}{M} + \frac{M-1}{M} zF(z)^2 \text{ when } w = y \\
0 &= \frac{M-1}{M} zF(z)^2 - F(z) + \frac{z}{M}
\end{aligned}$$

$$F(0) = 0$$

$$F(z) = \frac{M}{2(M-1)z}(1-\sqrt{1-4\frac{M-1}{M}z^2}) \text{ by quadratic formula }, z \neq 0$$

*Since $F(z) = F(x,y|z)$ and (by earlier lemma) $F(x,y|z) = F(x,w|z)F(w,y|z) \Rightarrow$

$$F(z) = F(x,y|z) = F(x,w_1|z)F(w_1,w_2|z)\cdots F(w_{k-1},y|z) = F(z)^k = F(z)^{d(x,y)}$$

□

Corollary 0.50. *For $m \geq 3$, simple random walks on T_m are transient.*

Lecture 7. September 23, 2009

Reversible Markov Chain

Definition 0.51. (X,P) is **reversible** if $\exists X \xrightarrow{m} (0,\infty)$ such that

$$a(x,y) = m(x)p(x,y) = m(y)p(y,x).$$

Example 0.52. Cayley graph with symmetric random walk and $m \equiv 1$.

Note: a in the above definition is called the **conductance** and $m(x) = \sum_y a(x,y) = \sum_y m(x)p(x,y)$ is the **total conductance**.

Oriented Graphs

(X,E) is an oriented graph with the property that $\vec{xy} \in E \Leftrightarrow a(x,y) > 0$ (i.e. we assume there is a reversible Markov chain with state space X and this last property). Think of functions on E as flows. The **resistance** of $e \in E$ is $r(e) \equiv \frac{1}{a(e^-,e^+)}$, where $e = \vec{e^-e^+}$. Let $N = (X,E,r)$ be the **network**. N is recurrent/transient if X is recurrent/transient.

Let

$$\nabla : \{\text{functions on } X\} \longrightarrow \{\text{functions on } E\} \text{ be given by } (\nabla f)(e) = \frac{f(e^+)-f(e^-)}{r(e)}$$

We can think of this as the gradient. The adjoint is:

$$(\nabla^* u)x = \frac{1}{m(x)}\Big(\sum_{e \text{ s.t.} e^+=x} u(e) - \sum_{e^-=x} u(e)\Big)$$

If you think about this as a flow, then this tells you how much is going out versus how much is coming in (i.e. the net loss).

Finite Energy Flow:

Definition 0.53. Let $x \in X, i_0 \in \mathbb{R}$. A "function u on E" (i.e. u is in $L^2(E,r)$) is a **finite energy flow** from x to ∞ with input i_0 at x if $\nabla^*(u)(y) = -\frac{i_0}{m(x)}\delta_x(y)$.

$l^2(X,m) = \{f : X \to \mathbb{R} \mid \sum f(x)^2 m(x) < \infty$ and $<f,g>= \sum f(x)g(x)m(x)\}$
$l^2(E,r) = \{u : E \to \mathbb{R} \mid \sum u(e)^2 r(e) < \infty\}$
Note: $||\nabla|| < \sqrt{2}, \nabla f(e) = \frac{f(e^+)-f(e^-)}{r(e)}, ||\nabla f(e)|| = \sum \frac{(f(e^+)-f(e^-))^2}{r(e)}$
Adjoint: If $\mathcal{H}_1 \xrightarrow{u} \mathcal{H}_2$ and $\mathcal{H}_1 \xleftarrow{u^*} \mathcal{H}_2$ such that $< uf, g >=< f, u^*g >$, then u^* is adjoint to u.

Lemma 0.54. ∇, ∇^* *are adjoint operators.*

Proof. Check that $< \nabla f, u >=< f, \nabla^* u >$

$$\begin{aligned}
\text{RHS} &=< f, \frac{1}{m(x)}\Big(\sum_{e^+=x} u(e) - \sum_{e^-=x} u(e)\Big) > \\
&= \sum_x \frac{1}{m(x)}\Big(\sum_{e^+=x} f(x)u(e) - \sum_{e^-=x} f(x)u(e)\Big)m(x) \\
&= \sum_e u(e)\big(f(e^+) - f(e^-)\big) \\
&= \sum u(e)\nabla f(e) \cdot r(e) \\
&= \text{LHS}
\end{aligned}$$

□

The Laplacian is $\mathcal{L} = -\nabla^*\nabla$

Proposition 0.55. $-\nabla\nabla = P - I$*, where* I *is the identity matrix and* P *is a matrix on* X *given by* $(Pf)(x) = \sum_y p(x,y)f(y)$.

Proof.

$$\begin{aligned}
(-\nabla^*\nabla f)(x) &= -\frac{1}{m(x)}\big(\sum_{e^+=x} (\nabla f)(e) - \sum^{e^-=x} (\nabla f)(e)\big) \\
&= \frac{1}{m(x)}\big(\sum_{e^+=x} \frac{f(e^-) - f(x)}{r(e)} + \sum_{e^-=x} \frac{f(e^+) - f(x)}{r(e)}\big) \\
&= \sum_{e^+=x} \frac{a(x,e^-)}{m(x)f(e^-)} - \frac{a(e^-,x)}{m(x)}f(x) + \sum_{e^-=x} \frac{a(x,e^+)}{m(x)f(e^+)} - \frac{a(e^+,x)}{m(x)}f(x) \\
&= \sum_{e^+=x} p(x,e^-)f(e^-) - p(e^-x)f(x) + \sum_{e^-=x} p(x,e^+)f(e^+) - p(x,e^+)f(x) \\
&= \sum_y p(x,y)f(y)
\end{aligned}$$

□

Definition 0.56. $\mathcal{D}(N) = \{f : X \longrightarrow \mathbb{R} \mid \text{s.t. } \nabla f \in l^2(E, r)\}$

Note: do NOT require $f \in l^2(X, m)$. Endow $\mathcal{D}(N)$ with the **Dirichlet sum norm**.

$$D(f) = <\nabla f, \nabla f> = \sum_e \frac{(f(e^+) - f(e^-))^2}{r(e)} = \frac{1}{2} \sum_{x,y \in X} (f(x) - f(y))^2 m(x) p(x, y)$$

Note that $D(f) = 0 \Rightarrow f$ is constant (P irreducible).
Fix $o \in X$. Define an inner product on $\mathcal{D}(N) \ni f, g$ as:

$$< f, g > = < \nabla f, \nabla g > + f(o)g(o)$$

Lecture 8. September 25, 2009

r is resistance in $L^2(E, r)$ and P is potential in (X, P).
Hilbert space: vector space with inner product that is complete

Lemma 0.57. *$\mathcal{D}(N)$ is a Hilbert space and*

(1) *changing the base point leads to equivalent norms*
(2) *convergence in $D(f)$ implies pointwise convergence*
(3) *$f \in \mathcal{D}(N) \Rightarrow \nabla\nabla^*$ is defined.*

.

Two norms $||\cdot||$ and $||\cdot||_a$ are equivalent if $\exists c$ such that $\forall f$, $\frac{1}{c}||f||_a \leq ||f|| \leq ||f||_a$.

Proof. Basic inequality. Let $x \in X, x \neq o$. Then there are edges $e_{i+1} = x_i x_{i+1}$ from o to x (between points x_i and x_{i+1}). Let $C_1(x) = \sum_{i=1}^n r(e_i)$. Then for $f \in \mathcal{D}(N)$,

$$\begin{aligned}
(f(x) - f(o))^2 &\leq [f(x) - f(x_{n-1} + f(x_{n-1}) - f(x_{n-2}) + \cdots - f(o)]^2 \\
&= \left(\sum_{i=1}^n \frac{(f(x_i) - f(x_{i-1}))}{\sqrt{r(e_i)}} \sqrt{r(e_i)} \right)^2 \\
&\leq \left(\sum_{i=1}^n \frac{(f(x_i) - f(x_{i-1}))^2}{r(e_i)} \right) \left(\sum_{i=1}^n r(e_i) \right) \text{ Cauchy-Schwartz} \\
&\leq \left(\sum_{i=1}^n r(e_i) \right) D(f)
\end{aligned}$$

□

$l_0(X)$ is all functions that are finitely supported on X and $\mathcal{D}_0(N) = \overline{l_0(X)}$ in $\mathcal{D}(N)$. D_{p^k} is the Markov process where p is replaced (in D_p) by p^k.

Lemma 0.58. *$D_{p^k}(f) \leq k^2 D(f), \forall f \in l_0(X)$ (or $\mathcal{D}_0(N)$).*

Proof. Use Cauchy-Schwartz □

Proposition 0.59. *If (X, P) is transient, then $G(\cdot, X) \in \mathcal{D}_0(N)$.*

Idea of Proof. Show that the Green's function can be approximated by finitely supported functions. Approximate G by something like a Green's functions from finite subsets $A \subset X$. Let $A \subset X$ be finite.

$$p_A(x, y) = \begin{cases} p(x, y) & \text{if } x, y \in A \\ 0 & \text{else} \end{cases} \quad \text{and } p_A^{(n)}(x, y) = \mathbb{P}_x[z_n = y, z_k \in A, \forall 0 \leq k \leq n]$$

This looks like a Markov process, but it is not quite one.

$$G_A(x,y|z) = \sum_{n=0}^{\infty} P_A^{(n)}(x,y)z^n, G_A(x,y) = G_A(x,y|1)$$

Warning: $G_A(x,y) \neq G(x,y)$ even if both $x, y \in A$.

Let $A, B \subset X$ be finite subsets containing x and let $f = G_B(\cdot, x), g = G_A(\cdot, x)$.

$$\begin{aligned} D(g-f) &= D\big(G_B(\cdot,x) - G_A(\cdot,x)\big) \\ &=< \nabla(G_B(\cdot,x) - G_A(\cdot,x)), \nabla(G_B(\cdot,x) - G_A(\cdot,x)) > \\ &=< \nabla G_B(\cdot,x), \nabla G_B(\cdot,x) > + < \nabla G_A(\cdot,x), \nabla G_A(\cdot,x) > -2 < \nabla G_B(\cdot,x), \nabla G_A(\cdot,x) > \\ &= m(x)G_B(x,x) + m(x)G_A(x,x) - 2m(x)G_B(x,x) \\ &= m(x)[G_A(x,x) - G_B(x,x)] \end{aligned}$$

Let $X = \cup A_n$, where the A_n are finite and increasing (i.e. $\cdots \subset A_n \subset A_{n+1} \subset \cdots$ is an exhaustion). Then $G_{A_n}(x,x)$ converges to $G(x,x)$ by monotone convergence. □

$$I_A(x,y) = \begin{cases} 1 & \text{if } x = y \in A \\ 0 & \text{else} \end{cases} \quad \text{and } \dagger\ (I_A - P_A)G_A = I_A$$

So that $((I_A - P_A)G_A)(\cdot, x) = \delta_x(\cdot)$. More explanation:

$$\begin{aligned} (P_A G_A)(y,x) &= \sum_z P_A(y,z)G_A(z,x) \\ &= \sum_z P_A(y,z) \sum_{n=0}^{\infty} P_A^{(n)}(z,x) \\ &= \sum_{n=0}^{\infty} \sum_z P_A(y,z)P_A^{(n)}(z,x) \\ &= \sum_{n=0}^{\infty} P_A^{(n+1)}(y,x) \\ &= G_A(y,x) - \delta_x(y) \end{aligned}$$

Also note that $(I_A(f))(y) = f(y)$ if $y \in A$ and $(I_A(G_A(\cdot,x))(y) = G_A(y,x)$ if $y \in A$

Lemma 0.60. *If $f \in l_0(X)$ and supp$f \subset A$, then $< \nabla f, \nabla G_A(\cdot, x) >= m(x)f(x)$.*

Proof.

$$\begin{aligned} < \nabla f, \nabla G_A(\cdot,x) > &=< f.\nabla^* \nabla G_A(\cdot, x0 > \\ &=< f, (I-P)G_A(\cdot,x) > \\ &=< f, (I-P_A)G_A(\cdot,x) > \text{ since outside } A, G_A \equiv 0 \\ &=< f, \delta_x > \\ &= m(x)f(x) \end{aligned}$$

□

Lecture 9. September 28, 2009

Definition 0.61. Let $A \subset X$. The **capacity** of A, cap$A$, is:

$$\text{cap}A = \inf\{D(f) \mid f \in l_0(X), f \equiv 1 \text{ on } A\} = \min\{D(f) \mid f \in \mathcal{D}_0, f \equiv 1 \text{ on } A\}$$

Basic fact from Hilbert space: Let E be closed, convex subset of a Hilbert space. Then $\exists! e \in E$ which minimizes the norm.

Theorem 0.62. *Let (X,P) be a reversible Markov chain. Then TFAE:*

1. *(X,P) is transient*
2. *for some (all) $x \in X, \exists$ a finite energy flow from x to ∞ with non-zero input*
3. *for some (all) $x \in X$, $capx > 0$*
4. *$f \equiv 1 \notin \mathcal{D}_0(N)$*

Proof. (1) $\Rightarrow$ (2):
Want $u \in L^2(E,r)$, $\nabla^* u = -\frac{i_0}{m(x)}$. Have $G(\cdot,x) \in D_0(N)$. Set $u = \frac{-i_0}{m(x)}\nabla G(\cdot,x)$.

$$\nabla^* u = \nabla^*\nabla G(\cdot,x) = \frac{-i_0}{m(x)}\delta_x$$

Assume (2) $\Rightarrow$ (3). Show (3) $\Leftrightarrow$ (4).
cap$x = 0 \Leftrightarrow \exists f \in \mathcal{D}_0(N)$ with $f(x)=1$ and $\mathcal{D}(f)=0$ (i.e. $f \equiv 1$) $\Leftrightarrow 1 \in \mathcal{D}_0(N)$.
(3) $\Rightarrow$ (1) "Make the Green's function converge." ☐

Idea: Do a funny approximation scheme using G_A's for a finite $A \subset X$.
Let $A \subset X$ finite and $f = \frac{G_A(\cdot,X)}{G_A(x,x)}, G_A(\cdot,x) = \sum_y p_A^{(n)}(x,y)$.

$$\text{cap}(x) \le d(f) = \frac{1}{G(x,x)^2}\langle \nabla G_A(\cdot,x), \nabla G_A(\cdot,x)\rangle = \frac{1}{G_A(x,x)^2} m(x) G_A(x,x) = \frac{m(x)}{G_A(x,x)}$$

Therefore $G_A(x,x) \le \frac{m(x)}{\text{cap}(x)}$.
Since this works for any finite A, let $A_n \subset A_{n+1} \subset \cdot$ be an exhaustion of X by finite sets. Then $G(x,x) \leftarrow G_A(x,x) \le \frac{m(x)}{\text{cap}(x)}$ so this is bounded and hence converges.

Corollary 0.63. *If a subnetwork N is transient, then the network is transient.*

Proof. Use (4). ☐

Shorting:
Let (X,P) be an irreducible Markov chain, N a network, and $1_{X_i} = \chi_{X_i}$ characteristic functions. $X = \sqcup_i X_i$ a partition with $1_{X_i} \in \mathcal{D}_0(N), \forall i$.
Shorted Network: Collapse X_i and set:

$$a'(X_i,X_j) = \begin{cases} \sum_{x\in X_i, y \in X_j} a(x,y) & i \ne j \\ 0 & i = j \end{cases} \text{ and } m'(i) = \sum_j a'(i,j) = \mathcal{D}(1_{X_i}) < \infty$$

Theorem 0.64. *Let (X,P) be a reversible Markov chain and X' a shorted network. If (X',P') is recurrent then so is (X,P).*

Proof. For $f \in \mathcal{D}(N')$, lift f to N by $\overline{f}(x) = f(i), \forall x \in X_i$. Then

$$\begin{aligned} D_N(\overline{f}) &= \frac{1}{2}\sum_{x,y\in X}\big(\overline{f}(x) - \overline{f}(\overline{y})\big)a(x,y) \\ &= \frac{1}{2}\sum_{i,j\in X'}\sum_{x\in X_i, y\in X_j}\big(f(i)-f(j)\big)^2 a(x,y) \\ &= \frac{1}{2}\sum_{i,j} \in X\big(f(i)-f(j)\big)^2 a'(i,j) \\ &= \mathcal{D}_{N'}(f) \end{aligned}$$

This last argumet works for $\mathcal{D}_0(N')$ just as well so $1 \in \mathcal{D}_0(N') \Rightarrow 1 \in \mathcal{D}_0(N)$. $\overline{f} = \sum f(i)1_{X_i} \in \mathcal{D}_0(N)$. 1 =constant on $X' \Rightarrow \exists f_n \in l_0(X')$ such that $f_n \ni 2$ in $\mathcal{D}_{N'}$ norm. ☐

Lecture 10. September 30, 2009

Nearest Neighbor Random Walk:
$X = \mathbb{N}$, edges= $\{[u, u+1]\}$
Transition probabilities: $p(m,u) = 0$ if $|m-n| \geq 2$, $p(m,m+1), p(m,m-1) > 0$ and $p(m,m) \geq 0 \Rightarrow$ reversible.
$a(x,y) = m(x)pp(x,y) = m(y)p(y,x)$ and $m(k)p(k,k-1) = m(k-1)p(k-1,k)$

$$m(k) = m(k-1)\frac{p(k-1,k)}{p(k,k-1)} = \cdots = \frac{p(0,1)p(1,2)\cdots p(k-1,k)}{p(1,0)p(2,1)\cdots p(k,k-1)} = m(k)$$

$e_k \equiv [k-1,k]$ and $r(e_k) = \frac{1}{a(k-1,k)} = \frac{p(k-1,k-2)\cdots p(1,0)}{p(0,1)\cdots p(k-1,k)}$.
The only flow from 0 to ∞ with input 1 is $u \equiv 1$.

$$< u, u >= \sum_{k=1}^{\infty} 1 \cdot 1 \cdot r(e_k) = \sum_{k=1}^{\infty} r(e_k) \begin{cases} = \infty & \text{if recurrent} \\ < \infty & \text{if transient} \end{cases}$$

Shortenings:
(X,P) rev Markov chain, N-network. $X = \cup_i x_i$ with $\chi_{x_i} = 1_{x_i} \in \mathcal{D}_0(N)$. Collapse X_i. Get $X' = \{i\}$ and $a'(i,j) = \begin{cases} \sum_{x\in X_i, y\in X_j} a(x,y) & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$.

Proposition 0.65. *If the shortening X' is recurrent then X is recurrent.*

Proof. If X' is recurrent, then $1_{X'} \in \mathcal{D}_0(N')$. $X' \xrightarrow{f}$ lifts to $X \xrightarrow{\overline{f}} \mathbb{R}$. We want to show that $1_X = \overline{1_{X'}}$. Why is $1_x \in \mathcal{D}_0(N)$? Approximate $1_{X'}$ by $f_h \in l_0(X')$. Note that $\overline{f}_h \xrightarrow[\text{in } D]{} \overline{1}_{X'} = 1_X$, where $\overline{f}_h \in \mathcal{D}_0(N)$. $D(\overline{1}_{X'} - \overline{f}_h) = D(1_{X'} - f_h) \to 0$. ☐

Nash-Williams Recurrence Criterion:
Suppose $X' = \mathbb{N}$ and $a'(i.j) = 0$ if $|i-j| \geq 2$. Then $\sum \frac{1}{a'(i-1,i)} = \infty \Rightarrow (X,P)$ ie recurrent.

Let Γ be a finitely generated group and S a generataing set. d_{word} is the word metric (depends on S). $d_{\text{word}}(1,\gamma) = d(1,\gamma)$ =the minimum size of a word in S that expresses γ. $d(a,b) = d(1,a^{-1}b)$. $\#B(1,n) \equiv V(n) =$"volume of a ball of size n."

Example 0.66. $G = \mathbb{Z}, S = \{\pm 1\}$. Then $V(n) = 2n+1$. If instead $S = \{\pm 1, \pm 2\}$, then $V(n) \sim 4n$.

Example 0.67. $G = F_2, S = \{a, b, a^{-1}, b^{-1}\}$. Then $V(n) = \sum_{k=1}^{n} 4 \cdot 3^{k-1} \sim 4 \cdot 3^{n-1} = Ce^{n \log 3}$, for some constant C. If instead $S = \{a^{\pm 1}, b^{\pm 1}, a^{\pm 2}, b^{\pm 2}\}$, then $V(n) \sim C' \cdot 9^{n-1} = C'e^{n \log 9}$, for some constant C'.

Exponential growth depends on S.

Lemma 0.68. *If growth is exponential for S so it is for S^*.*

Given Γ, is there a lower bound for the exponent of exponential growth? Typically yes, but not always.

Example 0.69. $G = \mathbb{Z}^2, S = \{\pm e_1, \pm e_2\}$. Then a ball of volume n roughly looks like a square and $V(n) \sim Cn^2$.

Basic Observation: In $\mathbb{Z}^2$, for all S growth is quadratic. In $\mathbb{Z}^n$, for all S growth is of order n.

Example 0.70. Heisenberg Group:

$$\text{Heis} = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$$

$$\text{Heis}_{\mathbb{Z}} = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{Z} \right\}$$

$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $c = [a, b] = aba^{-1}b^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

δ_n-dilation:

$$\delta_n \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & na & n^2c \\ 0 & 1 & nb \\ 0 & 0 & 1 \end{pmatrix}$$

$\delta_n([a, b]) = \delta_n(c) = n^2c$ and $\delta_n([a, b]) = [\delta_n a, \delta_n b] = n^2c$. So $V(n) \sim cn^4$.

Lecture 11. October 2, 2009

Let M be a compact Riemannian manifold (with no boundary) with Riemannian metric g. Let $\widetilde{M}$ be the universal cover of M and $\Gamma = \pi_1(M)$ be finitely generated. Γ acts on $\widetilde{M}$ by deck transformations. So $\forall p \in M$, $g_p(v, w)$ is an inner product on M. This leads to a metric $\widetilde{g}$ on $\widetilde{M}$ and an inner product $\widetilde{g}_{\widetilde{p}}(\widetilde{v}, \widetilde{w}) = g_p(v, w)$ on $\widetilde{M}$.
Volume growth of $\widetilde{M}$: Let $B_p(r) = \{y \mid d(p, y) \leq r\}$. Then $\text{vol}(B_p(r)) = \widetilde{V}(r)$. This does not depend on the particular p.

Proposition 0.71. *$\widetilde{V}(r)$ and $V_\Gamma(r)$ have "similiar" growth:*

- *Either both have exponential growth*
- *Or both have polynomial growth of the same degree*
- *Or not*

Corollary 0.72. *If M is closed Riemannian manifold with $k < 0$ ($k \leq 0$ except for flat manifolds), then $\pi_1(M)$ has exponential growth.*

Definition 0.73. Let $(X, d_X), (Y, d_Y)$ be metric spaces. A map $X \xrightarrow{\varphi} Y$ is a **quasi-isometry** if $\exists A, B > 0$ such that:

(1) $\forall x_1, x_2 \in X, \quad \frac{1}{A} d(x_1, x_2) - B \leq d(\varphi(x_1), \varphi(x_2)) \leq Ad(x_1, x_2) + B$
(2) $\forall y \in Y, \exists x \in X$ such that $d(y, \varphi(x)) < B$

Lemma 0.74 (Milnor, Svarc)**.** *Let (M, g) be a Riemannian manifold. Let $\Gamma = \pi_1(M)$ with word metric for some finite set of generators and let $\widetilde{M}$ have lifted Riemannian metric. Then Γ and $\widetilde{M}$ are quasi-isometric: $\Gamma \longrightarrow \widetilde{M}$ is given by $\gamma \mapsto \gamma \circ 0$*

Lemma 0.75. *For $\Gamma = \pi_1(M)$, let S, T both be finite generating sets and let d_S, d_T be the word metrics for S, T (respectively). Then $id : (\Gamma, d_S) \to (\Gamma, d_T)$ is a quasi-isometry.*

Theorem 0.76 (Tits alternative)**.** *Let $\Gamma \subset GL(n, \mathbb{R})$ be finitely generated. Then either $\Gamma \supset F_2$ or Γ is solvable.*

Lemma 0.77. *If $\Gamma \supset F_2$, then Γ has exponential growth.*

Sketch of Proof. Let (wlog) S be the generating set of Γ and take it to contain a, b such that $< a, b >= F_2$. □

Lemma 0.78 (Milnor, Wolf)**.** *Let $\Gamma \subset GL(n, \mathbb{R})$ be solvable. Then either Γ has exponential growth or Γ has a nilpotent subgroup of finite index.*

Idea of Proof. Prime example: Suppose $\exists \varphi \in \mathrm{GL}(k, \mathbb{Z})$, $\mathbb{Z}^k \xrightarrow{\varphi} \mathbb{Z}^k$ an automorphism, $\Gamma = \mathbb{Z} \rtimes \mathbb{Z}^k$ (semi-direct product) given by $(m, z) \cdot (n, t) = (m + n, \varphi^n(z) + t)$. □

Dichotomoy: Either all eigenvalues of φ have absolute value 1 or $\exists$ a value λ of φ with $|\lambda| > 1$. For the first situation, either the image of φ^k is dense (not possible) or finite $\Rightarrow \varphi^l = id$ for some l.

Theorem 0.79. *Let Γ be finite generated subset of $GL(n, \mathbb{R})$. Then either Γ has exponential growth or $\Gamma \supset N$ is a nilpotent subgroup of finite index and it has polynomial growth.*

Theorem 0.80 (Gromov)**.** *Γ has polynomial growth $\Leftrightarrow \Gamma \supset N$, where N is nilpotent of finite index.*

Warnng: $\exists\Gamma$ finitely generated with non-polynomial growth but non-exponential growth.

Lecture 12. October 5, 2009

Application of Shortocomings and Volume Growth:
Let Γ be a finitely generate group with volume growth. $V(n) \leq cn^2$ for some constant c (so Γ has at most quadratic growth). Then the simple random walk on Γ is recurrent.

Corollary 0.81 (Gromov)**.** *Γ must contain $1, \mathbb{Z}$, or $\mathbb{Z}^2$ as a subgroup of finite index to have quadratic growth.*

Proof of Application. Partition Γ into spheres $S_n = S(n) = \{\gamma \mid d(\gamma, 1) = n\}$. Consider shorening attached, i.e. collapse $S(n)$ to n for all n.

$$a'(n, n+1) \leq \sum_{x \in S_n, y \in S_{n+1}} 1 \leq d\#S_n = d(V(n) - V(n-1)), \text{ where } d\# \text{ generators}$$

Hence,

$$\sum_{k=n}^{2n} \frac{1}{a'(k, k+1)} \geq^* \frac{n^2}{\sum_{k=n}^{2n} a'(k, k+1)} \geq \frac{1}{d} \frac{n^2}{V(2n)} \geq \frac{1}{dc} > 0$$

(*) Use Cauchy-Schwarz inequality.
$\Rightarrow \sum_0^\infty \frac{1}{a'(k,k+1)} = \infty \Rightarrow$(Nash-Williams) recurrence of shortening and hence of Γ.
Note: these are shortenings since the S_n are finite. □

Goal: Show that recurrence is quasi-isometry invariant by comparing Markov chains.
Let (X, P) be a Markov chain, action on measure on X, and ν be a measure on X.

$$(\nu^p)(y) = \sum_{x \in X} \nu(x) p(x, y) \text{ (assume finite)}$$

$f : X \to \mathbb{R} \Rightarrow Pf = \sum_z f(z) p(y, z)$

Definition 0.82. Let (X, P) be a Markov chain on X and ν a measure on X. Then ν is **invariant** if $\nu^p = \nu$. If $\nu^p \leq \nu$ (pointwise) then ν is **excessive**.

Remark 0.83. If ν is excessive and $\nu(x) = 0$ for some x, then $\nu \equiv 0$ (irreducibility).

Proposition 0.84. *(X, P) is recurrent $\Leftrightarrow \exists$ invariant measure $\nu \neq 0$ such that all excessive measures are multiples of ν.*

Idea of Proof. Set $Q(x, y) = \frac{\nu(y)}{\nu(x)} P(y, x)$ for ν P-invariant. Then Q is a Markov chain:

$$\sum_y Q(x, y) = \frac{1}{\nu(x)} \sum_y P(y, x) = \frac{1}{\nu(x)} \nu(x) = 1$$

Given (X, P) recurrent, how to constuct an invariant measure ν: Fix a base point σ.

$$\nu(x) = \text{ expected number of visits to state } x \text{ before returning to } \sigma$$

We need recurrence for this to be finite. □

Definition 0.85. Let $\mathcal{H}$ be a Hilbert space. $T : \mathcal{H} \to \mathcal{H}$ is a **contraction** if $< Tf, f > \leq < f.f >, \forall f$.

Lemma 0.86. *If ν is an excessive measure for (X, P), then P is a contraction on $l^2(X, \nu)$.*

Proof.

$$\begin{aligned}
< Pf, f > &= \sum_{x \in X} (Pf)(x) f(x) \cdot \nu(x) \\
&= \sum_{x \in X} \sum_{y \in X} \nu(x) f(x) f(y) P(x, y) \\
&\leq \sum_{x, y \in X} \nu(x) P(x, y) \left(\frac{f(x)^2}{2} + \frac{f(y)^2}{2} \right) \\
&\leq \sum_{x \in X} \frac{f(x)^2}{2} \nu(x) + \sum_{y \in X} \frac{f(y)^2}{2} \nu(y) \text{ by (*)} \\
&= \frac{< f, f >}{2} + \frac{< f, f >}{2} \\
&= < f, f >
\end{aligned}$$

(*) Since $\sum_{x \in X} \nu(x) P(x, y) \leq \nu(y)$.
Hence P is a contraction. □

Note: P^* is the adjoint of P and is given by: $P^*(x, y) = \frac{\nu(y)}{\nu(x)} P(x, y)$.

Lemma 0.87 (Hilbert Space)**.** *Let $\mathcal{H}$ be a Hilbert space with inner product $\langle , \rangle$. Let $T_1, T_2 : \mathcal{H} \to \mathcal{H}$ be invertible linear operators such that:*

1. *T_1 is self-adjoint*
2. *$\langle T_2 f, f \rangle \geq \langle T_1 f, f \rangle \geq 0, \forall f \in \mathcal{H}$*

Then $\forall f \in \mathcal{H}$, $\langle T_1^{-1} f, f \rangle \geq \langle T_2^{-1} f, f \rangle$.

Proof. Define a new inner product: $\langle \langle f, g >>= \langle f, T_1 g \rangle$.

$$\langle \langle g, f \rangle \rangle = \langle g, T_1 f \rangle =^* \langle T_1 f, g \rangle = \langle g, T_1 f \rangle = \langle \langle f, g \rangle \rangle$$

(*) Since T_1 is self-adjoint.
Use Cauchy-Schwartz on $\langle \langle , \rangle \rangle$:

$$\dagger \ \langle f, T_1 g \rangle^2 = \langle \langle f, g \rangle \rangle^2 \leq \langle \langle f, f \rangle \rangle \langle \langle g, g \rangle \rangle = \langle f, T_1 f \rangle \langle g, T_1 g \rangle$$

Hence,

$$\begin{aligned}
\langle T_2^{-1} f, f \rangle^2 &= \langle \langle T_2^{-1} f, T_1 T_1^{-1} f \rangle \\
&\leq^\dagger \langle T_2^{-1} f, T_1 T_2^{-1} f \rangle \cdot \langle T_1^{-1} f \rangle \\
&\leq \langle T_2^{-1} f, T_2 T_2^{-1} f \rangle \cdot \langle T_1^{-1} f, f \rangle \\
&= \langle T_2^{-1} f, f \rangle \cdot \langle T_1^{-1} f, f \rangle
\end{aligned}$$

□

Lecture 13. October 7, 2009

We are on our way to understand quasi-isometries of currents.

Theorem 0.88. *Suppose P is irreducible, ν is an excessive measure, and Q is reversible with $m =$total conductance. Assume that $C \equiv \sup \frac{m(x)}{\nu(x)} < \infty$ and $\exists \epsilon_0 > 0$ such that $P \geq \epsilon_0 Q$ (point-wise). If (X, P) is recurrent, then (X, Q) is recurrent.*

Proof. Set-up Set $\overline{P} = \frac{1}{2}(I+P), u(x) \equiv \frac{m(x)}{\nu(x)}, \overline{Q} = (1-\frac{1}{2c}u)I + \frac{1}{2c}uQ$, where I is the identity. ν is excessive for $\overline{P}$ (easy calculation). $\overline{Q}$ is reversible with total conductance $= \nu$ since $\nu(x)\frac{m(x)}{\nu(x)}Q(x,y) = \nu(y)\frac{m(y)}{\nu(y)}Q(x,y)$, Q is reversible, and m is total conductance.

Step 1 of Proof: $\overline{P}$ recurrent $\Rightarrow \overline{Q}$ recurrent

Set $\epsilon = \min\{\frac{1}{2}, \epsilon_0\} \Rightarrow \overline{P} > \epsilon_1\overline{Q}$ (show):

$\frac{u}{c} \le 1$ so $\epsilon_1\frac{u}{2c}Q \le \epsilon_1\frac{1}{2}Q < \frac{1}{2}P$

$\epsilon_1(1-\frac{1}{2c}u)I \le \frac{1}{2}I$

So $\epsilon_1\overline{Q} = \epsilon_1(1-\frac{1}{2c}u)I + \frac{1}{2c}uQ < \frac{1}{2}(P+I) = \overline{P}$

In addition, $\frac{1}{1-\epsilon_1}(\overline{P}-\epsilon_1\overline{Q})$ is Markov with excessive measure ν. $\sum_y \overline{P}(x,y) - \epsilon_1\overline{Q}(x,y) = 1-\epsilon_1$. By last lecture,

$$< (\overline{P}-\epsilon_1\overline{Q})f, f >_\nu \le (1-\epsilon_1) < f,f >_\nu, \forall f \in L^2(x,\nu)$$

$\forall 0 \le z \le 1$, $\langle z(\overline{P}-\epsilon_1\overline{Q})f,f\rangle_\nu \le \langle (1-\epsilon_1)I(F),f\rangle_\nu$ $Rightarrow \langle (I-z\overline{P})f,f\rangle_\nu \ge \langle \epsilon_1(I-z\overline{Q})f,f\rangle_\nu$ †

$z\overline{P} - z\epsilon_1\overline{Q} \le (1-\epsilon_1)I, \epsilon_1 I - z\epsilon_1\overline{Q} \le I - z\overline{P}$

Let $T_1 = \epsilon_1(I-z\overline{Q})$ and $T_2 = I - z\overline{P}$. $\overline{P},\overline{Q}$ are contactions on $l^2(X,\nu)$. Recall $(1-\alpha)^{-1} = \sum_{n=0}^\infty \alpha^n$ and $G_{\overline{P}}(x,y|z) = \sum_{n=0}^\infty \overline{P}^{(n)}(x,y)z^n = \sum_{n=0}^\infty (z\overline{P})^n(x,y)$.

$$(I-z\overline{P})^{-1}f)(x) = \sum_y G_{\overline{P}}(x,y|z)f(y)$$

$$((I-z\overline{Q})^{-1}f)(x) = \sum_y G_{\overline{Q}}(x,y|z)f(y)$$

Let $f = \delta_x$. $G_{\overline{P}}(x,x|z) \le \frac{1}{\epsilon_1}G_{\overline{Q}}(x,x|z)$. By †:

$$G_{\overline{P}}(x,x|z) = ((I-z\overline{P})^{-1}\delta_x)(x)$$

$$\frac{1}{\epsilon_1}G_{\overline{Q}}(x,x|z) = ((I-z\overline{Q})^{-1}\delta_x)(x)$$

Recall: Let L be a Hilbert space, T_1, T_2 are invariant and T_1 self-adjoint. So

$$\langle T_2 f, f\rangle \ge \langle T_1 f, f\rangle, \forall f \Rightarrow \langle T_1^{-1}f,f\rangle \ge \langle T_2^{-1}f,f\rangle$$

Step 2 of Proof: P recurrent $\Rightarrow \overline{P}$ recurrent

Calculation: $G_P(x,x|\frac{z}{2-z}) = (1-\frac{z}{2})G_{\overline{P}}(x,x|z)$ so for $z=1, G_p(x,x) = \frac{1}{2}G_{\overline{P}}(x,x)$. Indeed:

$(2-z)\sum_{n=0}^\infty P^{(n)}(\cdot,x)\frac{z^n}{(2-z)^n} - zP\sum_{n=0}^\infty P^{(n)}(\cdot,x)\frac{z^n}{(2-z)^n} =$

$\sum_{n=1}^\infty P^{(n)}(\cdot,x)\frac{z^n}{(2-z)^{n-1}} + (2-z)\delta_x(\cdot) - \sum_{n=0}^\infty P\cdot P^{(n)}(\cdot,x)\frac{z^{n+1}}{(2-z)^n} =$

$(2-z)\delta_x =$

$(2-z)G_P(\cdot,x|\frac{z}{2-z}) - zPG_P(\cdot,x|\frac{z}{2-z}) =$

$(2-z-zP)G_P(\cdot,x|\frac{z}{2-z})$

And $G_p(\cdot,x|\frac{z}{2-z}) = \frac{2-z}{2-z-zP}\delta_x = \frac{1-\frac{z}{2}}{1-z\overline{P}} = (1-\frac{z}{2})G_{\overline{P}}(x,x|z)$.

Step 3 of Proof: $\overline{Q}$ recurrent $\Rightarrow Q$ recurrent

$$D_{\overline{Q}}(f) = \frac{1}{2C}D_Q(f)$$

$$D_Q(f) = \frac{1}{2}\sum_{x,y}\big(f(x)-f(y)\big)^2\nu(x)\overline{q}(x,y)$$

$$= \frac{1}{2}\sum_{x,y}\big(f(x)-f(y)\big)^2\nu(x)\frac{u(x)}{2C}q(x,y)$$

$$= \frac{1}{2C}D_Q(f)$$

where the second equality for $D_Q(f)$ follows since $\delta_x(y) = 0$ unless $x = y$, in which case $f(x) - f(y) = 0$. □

Lecture 14. October 9, 2009

Remark 0.89. On existence of invariant measures ($\nu^p = \nu$): $\nu^p(y) = \sum_{x \in X} \nu(x)p(x,y)$
Suppose (X, P) is reversible. Conductance $a(x,y) = m(x)p(x,y) = m(y)p(y,x)$. Total conductance

$$m(x) = \sum_y a(x,y) = \sum_y m(y)p(x,yy) = m^D(x)$$

m is an invariant measure.

Definition 0.90. A Markov chain (X, P) is **uniformly irreducible** if $\exists K, \epsilon_0 > 0$ such that $\forall x \sim y$ (i.e. $p(x,y) \neq 0$) $p^k(x,y) \geq \epsilon_0$ for some $k < K$.

Let X be a graph with bounded geometry (i.e. a bounded number of edges at each vertex). Let D be the Dirichet norm for a simple random walk. Let D_P be the Dirichlet norm for the random walk for the Markov chain P.

Theorem 0.91. *Let (X, P) be a Markov chain with a P-transition matrix of a uniformly irreducible random walk on X with excessive measure ν such that $\inf \nu(x) > 0$. If (X, P) is recurrent, then simple random walks on X are recurrrent. Moreover, if P is reversible, $m = \nu =$total conductance, then $\exists \epsilon_1 > 0$ such that $D_p(f) \geq \epsilon_1 D(f)$.*

Proof. Let K, ϵ_0 as above and set $\overline{P} = \frac{I+P}{2}, \hat{P} = \overline{P}K$. We already know that $G_{\overline{P}}(x,x|z) = \frac{z}{2-z} G_p\big(x,x|\frac{z}{2-z}\big)$.So P recurrent $\Leftrightarrow$ $\overline{P}$ recurrent. $\overline{P}$ recurrent $\Rightarrow$ $\sum_{n=0}^{i} nfty \overline{p}^i(x,x) = \sum_{\alpha=0}^{k-1} \overline{p}^{kj+\alpha}(x,x)$ diverges. Therefore, for some $1 \leq \alpha \leq K, \sum_{n=0}^{\infty} \overline{p}^{kn_\alpha}(x,x) = 0$ and $\sum_{n=0}^{\infty} \hat{p}^{(n)}(x,x) \geq \frac{1}{2^{k-\alpha}} \sum_{n=0}^{\infty} \overline{p}^{nk+\alpha}(x,x)$ since $\overline{P} \geq \frac{1}{2} I$. Then $\overline{p}^l(x,x) \geq \frac{1}{2} \overline{p}^{l+\alpha}(x,x)$. so $\overline{P}$ recurrent $\Rightarrow \hat{P}$ recurrent. Note: ν is $\hat{P}$ excessive.
Claim: $\hat{P}$ dominates a multiple of the simple random walk
Note: in simple random walk, x, y adjacent $\Rightarrow Q(x,y) = \frac{1}{\deg(x)}$.
$x \sim y, \hat{p}(x,y) \geq \frac{\epsilon_0}{2^k} \geq \frac{\epsilon}{2^k} \frac{1}{\deg x}$. Apply the comparison theorem to $\hat{P}$ and simple random walk with $m(x) = \deg(x)$. Check the rest of this computation. □

Definition 0.92. A Markov chain (X, P) has **bounded range** if $\exists D > 0$ such that if $d(x,y) > D \Rightarrow P(x,y) = 0$.

Theorem 0.93. *Let (X, P) be a Markov chain of reversible random walk on X with bounded range and with $\sup_x m(x) < \infty$. Then $\exists \epsilon_2 > 0$ such that $D(f) \geq \epsilon_2 D_P(f)$.*

Thus if a simple random walk is recurrent, so is (X, P).
Motivation: For the moment, assume the conclusion about ϵ_2.
Recall:

- $\text{cap}(x) = \inf\{D(f) \mid f \in D_0(X), f(x) = 1\}$
- transience $\Leftrightarrow$ $\text{cap}(x) > 0$ for some x

Simple random walk recurrent $\Rightarrow$ $\text{cap}_{\text{simple}}(x) = 0 \Rightarrow \text{cap}_P(x) = 0 \Rightarrow \text{cap}(X,P)$ is recurrent.

Lecture 15. October 12, 2009

Proof. Let E be the edges for X. For $x, y \in X$, denote by

$\pi(x,y) = \{$geodesics from x to $y\}$ and $\pi_e(x,y) = \{$geodesics from x to y containing e

$|f(x)-f(y)|^2 \leq \frac{1}{|\pi(x,y)|}\sum_{\pi\in\pi(x,y)}\sum_{e\in\pi}(\nabla f(e))^2 d(x,y)$ by Cauchy-Schwartz. For $f \in l_0(x)$,

$$\begin{aligned} D_P(f) &= \frac{1}{2}\sum_{x,y\in X} a(x,y)\big(f(y)-f(x)\big)^2 \\ &\leq \frac{1}{2}\sum_{x,y} a(x,y)\frac{1}{\pi(x,y)}\sum_{\pi\in\pi(x,y)}\sum_{e\in\pi}(\nabla f(e))^2 d(x,y) \\ &= \sum_{e\in E}(\nabla f(e))^2\phi(e) \end{aligned}$$

where $\phi(e) = \frac{1}{2}\sum_{x,y\in X} m(x)p(x,y)d(x,y)\frac{\pi_e(x,y)}{\pi(x,y)}$ and $a(x,y) = m(x)p(x,y)$. Then $D_p(f) \leq \sup_{e\in E}\phi(e)D(f)$.
Claim: ϕ is bounded above Estimate $\phi(e)$: Let $M \geq$ vertex degrees, R be a bound for range P. and $e \in E$. Suppose $x,y \in X$ such that $d(x,y) \leq R$ and $\pi_e(x,y) \neq \phi \Rightarrow x,y$ are of distance $\leq R$ from the closest edge of e. There are at most M^R y's and M^r x's $\Rightarrow M^{2R}$ such $(x,y) \Rightarrow$

$$\phi(e) \leq \frac{R}{2}(\sup_{z\in X} m(z))\sum_{x,y\in X,d(x,y)\leq R}\frac{\pi_e(x,y)}{\pi(x,y)} \leq \frac{R}{2}\sup_{z\in X} m(z)M^{2R}$$

□

Corollary 0.94. *Let $X^{(k)}$ be a graph with the same vertices as X and an edge between x,y if $d(x,y) \leq k$. Then $\epsilon_2 > 0$ such that $D_{X^{(k)}}(f) \geq D_X(f)\epsilon_2$.*

If $X \xrightarrow{\varphi} X'$ quasi-isometry, then $\exists$ a "rough" inverse ψ.

Lemma 0.95. *ψ is a quasi-isometry.*

$\psi \circ \varphi$ is bounded distance from the identity.

Theorem 0.96. *Suppose X, X' are quasi-isometric graphs with bounded degrees of vertices. Then $\exists\epsilon_3 > 0$ such that $D_{X'}(f) \geq \epsilon_3 D_X(f\circ\varphi)$, where $X \xrightarrow{\varphi} X'$ is a quasi-isometry.*

Corollary 0.97. *If X is recurrent, so is X'.*

Proof. Consider $\varphi(X) \subset X'$ and define graph structure of $\varphi(X)$ by:

$$x',y' \in \varphi(X) \text{ then } x' \sim y' \text{ if } x' = \varphi(x), y' = \varphi(y) \text{ and } x \sim y \text{ in } X$$

$\varphi(X)$ is a network with $a'(x',y') = \#$edges from x to y in X with $\varphi(x) = x', \varphi(y) = y'$ (i.e. we are shortening X with partition of it induces $\varphi^{-1}(y'), y' \in X'$).
Claim: $a'(x',y') \leq (M^{AB+1})^2$,
where $\frac{d(x,y)}{A} - B \leq \varphi(x_1,x_2) \leq Ad(x,y)_B$ and $M \geq \deg x, \forall x \in X$.
Proof: If $\varphi(x) = x' = \varphi(y) \Rightarrow \frac{d(x,y)}{A} - B \leq d(\varphi(x),\varphi(y)) = 0 \Rightarrow d(x,y) \leq BA$.
Let $f \in l_0(X')$, $D_X(f\circ\varphi) \leq (M^{AB+1})^2 \cdot D_{P(X)}(f)$, where $P(X)$ is the graph defined above. Now if $x' \sim y'$ in $\varphi(X)$, then

$$d_{X'}(x',y') = d_{X'}(\varphi(x),\varphi(y)) \leq Ad(x,y) + B = A + B \equiv K$$

$x,y \in X$ such that $\varphi(x) = x', \varphi(y) = y', d(x,y) \leq 1 \Rightarrow \varphi(X)$ is a subgraph of $(X')^{(K)} \Rightarrow D_{\varphi(X)}(f) \leq D_{(X')^{(k)}}(f) \leq \epsilon_2 D_{X'}(f)$. □

Lecture 16. October 14, 2009

Brief Summary:

(1) Recurrence of various random walks is equivalent
(2) A simple random walk is recurrent if the volume growth of the group is at most quadratic (using shortenings and Nash-Williams)

(3) Volume growth at least cubic implies transient
(4) Γ finitely generated is recurrent (or simple or any other "reasonable" random walk) $\Leftrightarrow \Gamma$ is almost $1, \mathbb{Z}, \mathbb{Z}^2$s

Definition 0.98. Γ is **almost** A means that there is a subgroup of finite index in Γ, which is A.

Proof of (4). Gromov's polynomial growth theorem implies that growth is either at most quadratic or at least cubic. If growth is quadratic, then Γ is nilpotent and (Bass) Γ is almost $1, \mathbb{Z}, \mathbb{Z}^2$. □

Example 0.99. If something is quasi-isometric to the plane, then it is recurrent since the plane is recurrent.

Definition 0.100. Let (X, P) be a Markov chain. Let

$$\sigma_x(n) = \mathbb{P}_x\big[d(z,x)) = n\big] = \sum_{y, d(x,y)=n} p(x,y)$$

The k**th moment** is $M_k(x) = \sum_n n^k \sigma_x(n)$. **Exponential moment** of order $< \infty$ if $\sup_x \sum_n e^{c_n} \sigma_x(n) < \infty$. $M_k(\mu) = \sup_{x \in X} M_k(x)$.

The kth moment could be finite or infinite.
Let Γ be a finitely generated group, μ irreducible probability measure on Γ. Counting meausre (invariant).

Theorem 0.101. *A simple random walk on* Γ *is recurrent* $\Leftrightarrow$ *some (every) symmetric irreducible random walk* μ *on* P *with* $M_2(\mu) < \infty$ *induces a recurrent random walk.*

Let e be an edge.

Lemma 0.102. $\phi(e) = \frac{1}{2} \sum_{x,y \in X} p(x,y) d(x,y) \frac{\#\pi_e(x,y)}{\#\pi(x,y)}$ *is bounded.*

Proof. Note that $\mu(x^{-1}y) = p(x,y)$. Let e_0 be an edge in the Cayley graph. Given points $x, y \in X$ and an edge e_0 in X, we move everything by x^{-1} to $0, x^{-1}y \in X$ and an edge $x^{-1}(e_0)$. w corresponds to $x^{-1}y$.

$$\begin{aligned}
\phi(e_0) &= \frac{1}{2} \sum_{w \in \Gamma} \mu(w) d(0,w) \sum_{x \in \Gamma} \frac{\pi_{x^{-1}e_0}(0,w)}{\pi(0,w)} \\
&\leq \frac{1}{2} \sum_{w \in \Gamma} \mu(w) d(0,w) \sum_{e \text{ edge}} \frac{\#\pi_e(0,w)}{\#\pi(0,w)} \\
&= \frac{1}{2} \sum_{w \in \Gamma} \mu(w) \frac{d(0,w)}{\#\pi(0,w)} \sum_{\pi \in \pi(0,w)} \#\{e \mid e \in \pi\} \\
&= \frac{1}{2} \sum_{w \in \Gamma} \mu(w) \frac{d(0,w)^2}{\#\pi(0,w)} \#\pi(0,w) \\
&= \frac{1}{2} M_2(0)
\end{aligned}$$

The inequality arises because the group does not act transitively on the edges. □

Lemma 0.103. *Let* μ *be a probability measure on* Γ. *Decompose* μ *as* $\mu = \mu_1 + \mu_2$, *with* μ_1, μ_2 *both non-negative measures. Then* $\forall x \in \Gamma, \forall n \in \mathbb{N}$,

$$\mu^{(n)}(x) \leq \mu_1(\Gamma)^n + n||\mu_2||_\infty,$$

where $\mu_1(\Gamma)$ *is the total mass of* Γ *with repect to* μ_1 *and* $||\mu_2||_\infty = \sup_{\gamma \in \Gamma} \mu_2(\gamma)$.

Let $\lambda_k = \frac{1}{k^3 \log^2 k}$, for $k \geq 2$ and $\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k$. Let $V(k) = \#B(k)$, where $B(k)$ is the ball of radius k in Γ. Set $\mu(x) = \sum_{k=1}^{\infty} \frac{\lambda_k}{V(k)} \chi_{B(k)}(x)$, where χ is the characteristic function.

Proposition 0.104.

(1) $M_2(\mu) < \infty$

(2) *If* $V(k) \geq Ck^3$, *then* μ *is transient*

Proof. (1):

$$\begin{aligned} M_2(\mu) &= \sum_{k=1}^{\infty} \frac{\lambda_k}{V(k)} \sum_{x \in \Gamma} d(0,x)^2 \chi_{B(k)}(x) \\ &\leq \sum_{k=1}^{\infty} \frac{\lambda_k}{V(k)} V(k) k^2 \\ &= \sum_{k=1}^{\infty} \frac{1}{k \log^2 k} \\ &< \infty \end{aligned}$$

(2):
Let $m \geq 2$ be arbitrary and decompose μ as $\mu = \mu_1 + \mu_2$, where

$$\mu_1(x) = \sum_{k=1}^{m-1} \frac{\lambda_k}{V(k)} \chi_{B_k(x)} \text{ and } \mu_2 = \mu - \mu_1 = \sum_{k=m}^{\infty} \frac{\lambda_k}{V(k)} \chi_{B_k(x)}.$$

Let $s_m = \sum_{k=m}^{\infty} \lambda_k$, then $\mu_1(\Gamma) = 1 - s_m$.

$$||\mu_2||_\infty = \sum_{k=m}^{\infty} \frac{\lambda_k}{V(k)} \leq \frac{s_m}{V(m)} \leq \sum_{k=m}^{\infty} \frac{\lambda_k}{V(m)}$$

$\mu^{(n)}(1) \leq (1 - s_m)^n + n \frac{s_m}{V(m)}$, by a previous lemma.
Fact: $s_m m^2 \log^2 m \to \frac{1}{2} \Rightarrow$ for large $m, s_m \sim \frac{1}{2m^2 \log^2 m}$

$$(1 - s_m)^n = e^{n \log(1 - s_m)} \sim e^{n \log(1 - \frac{1}{2m^2 \log^2 m})} = e^{n \log(\frac{2m^2 \log^2 m - 1}{2m^2 \log^2 m})}$$

Therefore , $\mu^{(n)}(1) \leq e^{\frac{-c_1 n}{m^2 \log m}} + \frac{c_2 n}{m^5 \log^2 m}$.
$\forall n, \forall$ large m, set $m = m(n) = n^{\frac{2}{5}} \Rightarrow$ convergence of $\sum_{n=1}^{\infty} \mu^{(n)}(1) \Rightarrow$ transience. □

Proof of Theorem. ($\Leftarrow$) obvious. ($\Rightarrow$) similar to (recurrence of simple random walk $\Rightarrow$ recurrence of finite range random walk). Use the lemmas and propositions between the statement of the theorem and this to prove this direction. □

Lecture 17. October 16, 2009

Words on Gromov's proof of the polynomial growth:
Γ is polynomial growth. Look at Γ from very far away and rescale. Let d_Γ be a word metric and let $d_l = \frac{1}{l} d_\Gamma$ be a new metric. Let X be a metric space and $Y_n \subset X$ such that $\lim_n Y_n = Y$. Then $\lim_l(\Gamma, d_l) = X$. Show that:

(1) X is locally compact
(2) X is locally connected *(hint: replace* Γ *by the Cayley graph, which is locally connected)*
(3) The Hausdorff dimension of X is finite
(4) ***Theorem:*** *by Montgomery-Zippin* If X is locally compact, locally connected, finite dimensional metric space, then $\mathrm{Isom}(X)$ is a Lie group.
(5) $\Gamma \hookrightarrow \mathrm{Isom}(X)$

The first two steps are true for groups in general, but the third is not. The third step uses crucially polynomial growth. The fourth set is closely related to Hilbert's 5^{th} problem.

Lecture 18. October 21, 2009

Kleiner's proof of polynomial growth theorem.

Theorem 0.105. *Γ graph with bounded geometry and polynomial growth $\Rightarrow$ space of harmonic functions is finite dimensional.*

Main tool: Poincare inequality,

Theorem 0.106 (Korevic-Schoen)**.** *Let Γ be a group with X its Cayley graph. Then $\exists \mathcal{H}$ a Hilbert space with a free faithful action of Γ on $\mathcal{H}$ and a "harmonic map" $X \xrightarrow{\varphi} \mathcal{H}$ which is Γ-invariant.*

Isoperimetric Inequalities:
Riemannian context: $c\text{vol}(B_p(r))^{\frac{d-1}{d}} \leq \text{area}(S_p(r))$, where $S_p(r)$ is a sphere of radius r. This is d-dimensional isoperimetric inequality (polynomial growth).
Markov Process-Reversible
Let (X,P) reversible, $m(x)$ total conductance, $a(x,y)$ conductance, and N network. Think of m as a measure on X. Let $A \subset X$ and $D \subset E$ (where E are the edges).

$$m(A) = \sum_{x \in A} m(x), a(D) = \sum_{e \in D} a(e^-, e^+), \text{ and } \delta A = \{ \text{ edges } e \mid e^- \in A, e^+ \notin A\}$$

This is the boundary of the network (and it replaces the sphere). $a(\delta A)$ "$=$" surface area.

Definition 0.107. Suppose $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is non-dereasing. N satisfies a **ϕ-isomperimetric inequality** if $\exists \kappa > 0$ such that $\phi(m(A)) \leq \kappa a(\delta A)$. If this holds for the simple random walk, then say that the graph satsfies the ϕ-isomperimetric inequality.

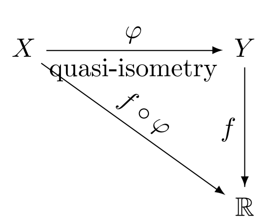
Special Cases:

(1) d-dimensional isomperimetric inequality: $\phi(t) = t^{\frac{d}{d-1}} = t^{1-\frac{1}{d}}$. $d = 1 \Leftrightarrow a(\delta A)$ is bounded below by some $\alpha > 0$.
(2) $\phi(t) = t$ strong isoperimetric inequality (this is $d = \infty$)

Goal: Relate isoperimetric inequality of quasi-isometric graphs.
Sobolev norm: $f : X \to \mathbb{R}$ function with P Markov chain.

$$S_P(f) = \sum_{e \in E} |\nabla f(e)| = \frac{1}{2} \sum_{x,y \in X} |f(x) - f(y)| a(x,y) \text{ and } ||f||_d = \left(\sum |f(x)|^d m(x)\right)^{\frac{1}{d}}$$

Proposition 0.108. *For $1 \leq d \leq \infty, (X,P)$ satisfies the d-dimensional isoperimetric inequality $\Leftrightarrow \forall f \in e_0(X), ||f||_{\frac{d}{d-1}} \leq \kappa S_P(f)$.*



Corollary 0.109. *If X and Y are quasi-isometric, then they satisfy the same d-dimensional isomperimetric inequality.*

Proof of Proposition. ($\Leftarrow$) easy: Let $X = \chi_A$.
($\Rightarrow$) It is sufficient to consider $|f|$ or $f \geq 0$. Note: $\big||a| - |b|\big| \leq |a - b|$. Rewrite $S_P(f)$:

$$\begin{aligned} S_P(f) &= \frac{1}{2} \sum_{x,y \in X} |f(x) - f(y)| a(x,y) \\ &= \sum_x \sum_{\{y | f(y) > f(x)\}} |f(y) - f(x)| a(x,y) \\ &= \sum_x \sum_{\{y | f(y) > f(x)\}} a(x,y) \int_0^\infty \chi_{[f(x), f(y)]}(t) dt \\ &= \int_0^\infty \Big(\sum_{\{x,y | f(x) \leq t < f(y)\}} a(x,y) \Big) dt \\ &= \int_0^\infty a(\delta\{y \mid f(y) > t\}) dt \end{aligned}$$

Case $d = 1$:
$\delta(\{y \mid f(y) > t\}) \neq \emptyset \Leftrightarrow 0 \leq t < ||f||_\infty$.

$$\int_0^\infty a(\delta(\{y \mid f(y) > t\}) = \int_0^{||f||_\infty} a(\delta\{y \mid f(y) > t\}) \geq \frac{1}{\kappa} ||f||_\infty$$

Case $d > 1$:
Set $\rho = \frac{d}{d-1}$.

$$\begin{aligned} S_P(f) &= \int_0^\infty a(\delta\{y \mid f(y) > t\}) dt \\ &\geq \frac{1}{\kappa} \int_0^\infty m(\{y \mid f(y) > t\})^{\frac{1}{\rho}} dt \\ &= \frac{1}{\kappa} \int_0^\infty F(t) dt \end{aligned}$$

where the inequality follows since $\kappa a(\delta\{y \mid f(y) > t\}) \geq m\{y \mid f(y) > t\}^{\frac{1}{\rho}}$. $F(t) = m(\{y \mid f(y) > t\})^{\frac{1}{\rho}}$ is non-increasing.

$$\rho (tF(t))^{\rho - 1} F(t) \leq \rho \Big(\int_0^t F(z) dz \Big)^{\rho - 1} F(t) = \frac{d}{dt} \Big(\int_0^t F(z) dz \Big)^\rho$$

Therefore,

$$\int_0^\infty \rho t^{\rho - 1} F(t)^\rho dt \leq \Big(\int_0^\infty F(t) dt \Big)^\rho$$

Therefore,

$$\begin{aligned} \kappa S_P(f) &\geq \int_0^\infty F(t) dt \\ &\geq \Big(\int_0^\infty F(t) dt \Big)^{\frac{\rho}{\rho}} \\ &\geq \Big(\int_0^\infty \rho t^{\rho - 1} F(t)^\rho dt \Big)^{\frac{1}{\rho}} \\ &= \Big(\int_0^\infty \rho t^{\rho - 1} m(\{y \mid f(y) > t\}) dt \Big)^{\frac{1}{\rho}} \end{aligned}$$

We want to be able to compare this to $||f||_\rho = ||f||_{\frac{d}{d-1}}$. Let $0 = t_0 < t_1 < \cdots < t_m$ be the values of f (in $l_0(X)$). Then

$$\begin{aligned}||f||_\rho^\rho &= \sum_{i=0}^{m} t_i^\rho\big(m(\{y \mid f(y) > t_{i-1}\}) - m(\{y \mid f(y) > t_i\}))\big)\\ &= \sum_{i=0}^{m} t_i^\rho m(\{y \mid f(y) = t_i\})\\ &= \sum_{i=0}^{m-1} (t_{i+1}^\rho - t_i^\rho) m(\{y \mid f(y) > t_i\})\\ &= \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \rho t^{\rho-1} m(\{y \mid f(y) > t\}) dt\\ &= \int_0^\infty \rho t^\rho m(\{y \mid f(y) > t\}) dt\end{aligned}$$

since $t_{i+1}^\rho - t_i^\rho = \int_{t_i}^{t_{i+1}} \rho t^{\rho-1} dt$. □

Lecture 19. October 23, 2009

Note: (X, P) satisfies a d-dimensional isoperimetric inequality $\Rightarrow V_p(n)) \geq cn^d$, for some $c > 0$. $V_p(n) \approx \sum_{r=0}^{n} m(S_P(r))$.

Amenabe Groups:

Goal: $\rho(X_\Gamma, \mu) = 1$, Cayley graph and spectural radius $\Leftrightarrow \Gamma$ is amenable.

Definition 0.110 (Definition/Theorem)**.** Let Γ be a finitely generated group. The following definitions/results are equivalent.

- **(0):** Γ is a **amenable** group
- **(1):** $\exists\Gamma$-variant mean on $l_\infty(\Gamma) = \{f : \Gamma \to \mathbb{R}$ bounded sup norm$\}$ *(a mean is $f \mapsto M(f)$ where essentially M is a finitelly additive measure)*
- **(2):** $\forall$ actions of Γ on a compact metric space X (by homeomorphisms) $\Gamma \to$Homeo(X) has a Γ-invariant probability measure
- **(3):** Fixed point property: $\rho : \Gamma \to U^*(B)$ and $C \subset B^*$ convex closed Γ-invariant set, then $\exists\Gamma$ fixed point in C
  Wrote this previous class, but there is a mistake: $\forall \rho : \Gamma \to U(B)$ and $\forall A \subset B^*$ convex and closed, $\exists$ fixed point for Γ in A *(Note: B is a Banach space, $U(B)$ is unitary (act by isometries on the Banach space), $B^* = \{$ continuous linear functionals $B \to \mathbb{R}\}$)*
- **(4):** *Folner's Property:* There are three versions:
  - **(a):** Folner sequence: $\{F_i\}$ is an exhaustion of Γ by finite sets such that $\forall g \in \Gamma, \lim_{i\to\infty} \frac{\#gF_i \Delta F_i}{\#F_i} = 0$. Γ satisfies Folner's property if $\exists$ a Folner sequence.
  - **(b):** Γ finitely generated, S a finite generating set of Γ. $\forall\epsilon > 0, \exists F \subset \Gamma$ such that $\frac{\#\partial F}{\#F} < \epsilon$.
  - **(c):** $\forall\epsilon > 0, \forall K \subset \Gamma$ finite, $\exists U \subset \Gamma$ finite such that $\forall k \in K$, $\frac{\#(U_k \Delta U)}{\#U} < \epsilon$.

Special Case: Let X be a compact metric space and Γ acts on X. Let $B = C(X) = \{$continuous functions on $X\}$ and so $B^* =$<probability meassure on X > by Riesz Representation theorem.

Lecture 20. October 26, 2009

Let Γ be discrete. Condition (1) of the previous definition is:

$l^\infty(\Gamma) \xrightarrow{\mu} \mathbb{R}, \mu(f) \geq 0, \mu(1) = 1 \Leftrightarrow \exists$ a finite additive probability "measure" μ on Γ which

is (left) Γ-invariant (i.e. $\mu : \{A \subset \Gamma\} \to [0,1], \mu(\Gamma) = 1$).
Examples:

(1) $\mathbb{Z}$ is amenable (show by property 4b)
(2) Solvable groups are amenable:
$\Gamma \supset \Gamma^1 = [\Gamma,\Gamma] \supset \Gamma^2 = [\Gamma^1,\Gamma^1] \supset \cdots \supset \Gamma^m \supset \Gamma^{m+1} = 1$. Γ^m is abelian, in a similar way to (1), Γ^m is amenable. Then repeat: Γ^{m-1}/Γ^m is abelian $\Rightarrow$ amenable.
(3) Subgroups of amenable groups are amenable. For $\Lambda \subset \Gamma$, Γ amenable, $\rho : \Lambda \to U(B) \rightsquigarrow \mathrm{Ind}_\Lambda^\Gamma(\rho) : \Gamma \to \{f : \Gamma \to B \mid f(\lambda\gamma) = \rho(\lambda)f(\gamma), \forall \lambda \in \Lambda\}$
(4) Short exact sequence: $1 \to \Gamma_1 \to \Gamma \to \Gamma_3 \to 1$. Then Γ is amenable $\Leftrightarrow$ Γ_1 and Γ_3 are amenable.
(5) Γ_1, Γ_2 amenable $\Rightarrow \Gamma_1 \times \Gamma_2$ amenable *(use fixed point property to prove)*
(6) Finitely generated groups with subexponential growth are amenable *(the converse is not true - there are amenable groups with exponential growth)*
(7) $\Gamma \subset \mathrm{GL}(n,\mathbb{R})$, Γ is amenable $\Leftrightarrow$ Γ is a finte extension of a solvable group
(8) A Lie group G is amenable $\Leftrightarrow$ G is a compact extension of a solvable group
(9) Free groups are not amenable

Proposition 0.111. *Let Γ be finitely generated. Γ is amenable $\Leftrightarrow \forall$ (or some) S a finite generating set of Γ, the Cayley graph of Γ with respect to S does not satisfy the strong isoperimetric inequality.*

Proof. ($\Rightarrow$) Use 4b in previous definition/theorem
($\Leftarrow$) Since S does not satisfy the isoperimetric inequality, there is no κ such that $\#\partial S) \geq \kappa \# F$. Therefore 4b is satisfied. □

Remark 0.112. The strong isoperimetric inequality implies that Γ has atleast exponential growth. Consider $V'(n)$"="area$S(n) \Rightarrow V'(n) \geq \kappa V(n)$.

Lecture 21. October 28, 2009

Groups:
(1) $\Rightarrow$ finite,compact $\Rightarrow \mathbb{Z}, \mathbb{Z}^2 \Rightarrow$ abelian $\Rightarrow$ nilpotent $\Rightarrow$ solvable
$\Rightarrow$ amenable $\Rightarrow F_2, \Gamma \subset SL(2,\mathbb{R})$ surface groups $\Rightarrow$
"negative curvature" Gromov hyperbolic groups $\Rightarrow SL(n,\mathbb{Z}), Mod(g,n), Out(F_n), Cat(0)$
Recall: $\rho(Q) = \overline{lim_{n\to\infty}} Q^{(n)}(x,y)^{\frac{1}{n}}$

Lemma 0.113. $Q = \big(Q(i,j)\big)_{i,j\in I}$ *symmetric, non-negative real matrix, irreducible* $\rho(Q) < \infty$*. Then Q acts on $l^2(I)$ as a bounded linear operator with operator norm:*

$$||Q|| = \sup_{0\neq f \in l^2(I)} \frac{||Q(f)||}{||f||} = \rho(Q)$$

Proof. $\langle , \rangle$ standard inner product on $l^2(I)$ and $f \in l^2(I)$.

$$\begin{aligned} \langle Q^{n+1}f, Q^{n+1}f\rangle^2 &= \langle Q^n f, Q^{n+2} f\rangle^2 \\ &\leq \langle Q^n f, Q^n f\rangle \langle Q^{n+2} f, Q^{n+2} f\rangle^2 \end{aligned}$$

$$\Rightarrow \quad \frac{\langle Q^{n+1}f, Q^{n+1}f\rangle}{\langle Q^n f, Q^n f\rangle} \leq \frac{\langle Q^{n+2}f, Q^{n+2}f\rangle}{\langle Q^n f, Q^n f\rangle} \xrightarrow[\text{test}]{\text{cf root}} \lim_{n\to\infty} \langle Q^n f, Q^n f\rangle^{\frac{1}{n}} \leq \rho(Q)^2$$

$\frac{\langle Qf, Qf\rangle}{\langle f,f\rangle} \leq \rho(Q)^2, \forall f \in l_0(I \Rightarrow \forall f \in l^2(I)$.
$\frac{||Q(f)||}{||f||} \leq \rho(Q)$. Check simple function like $f = \delta_x + \delta_y$. □

Theorem 0.114. (X,P) *reversible Markov chain. TFAE:*

(a): (X,P) *satisfies a strong isoperimetric inequality*
(b): $\exists \kappa > 0$ *such that* $||f||_2^2 \leq \kappa D_p(f), \forall f \in l_0(X)$

(c): *spectral radius* $\rho(P) < 1$
(d): *the Greens function* $G(x,y)$ *defines a bounded linear operator* $G : l^2(X,\mu) \to l^2(X,\mu)$ *via* $(Gf)(x) = \sum_y G(x,y)f(y)$.

Corollary 0.115 (Kestan)**.** Γ *is amenable* $\Leftrightarrow \rho(P) = 1$*, where* P *is a simple random walk on* Γ*.*

Recall: the spectral raduis is $\overline{\lim}_n P^{(n)}(x,y)^{\frac{1}{n}} < 1$, $P^{(n)}(1,1)$ decays exponentially fast.

Proof. $(a) \Rightarrow (b)$ Recall that $S_P(f) = \frac{1}{2}\sum_{x,y}|f(x) - f(y)|a(x,y)$. Also, recall that the strong isoperimetric inequality $\Rightarrow$ Sobolev inequality:

$$\exists \kappa > 0, \forall f \in l_0(X), \quad ||f||_1 \leq \kappa S_P(f)$$

$$\begin{aligned} ||f||_2^4 = ||f^2||_1^2 &\leq \kappa^2 S_P(f^2)^2 \\ &\leq \kappa^2 \left(\frac{1}{2}\sum_{x,y} a(x,y)\big(|f(x)| + |f(y)|\big)|f(x) - f(y)|\right)^2 \\ &\leq \kappa^2 D_P(f) \sum_{x,y\in X} a(x,y)\frac{\big(|f(x)| + |f(y)|\big)^2}{2} \\ &\leq \kappa^2 D_P(f) \sum_{x,y} a(x,y)\big(f(x)^2 + f(y)^2\big) \\ &\leq \kappa^2 D_P(f)||f||_2^2 \end{aligned}$$

$(b) \Rightarrow (a)$: Take $f = \chi_A$ for some $A \subset X$ finite. Then

$$|f||_1 = \sum f(x)m(x) = \sum f^2(x)m(x) = ||f||_2^2 \leq \kappa D_P(f) =^? \kappa S_P(f)$$

$(b) \Rightarrow (c)$: Recall the Laplacian $\mathcal{L} = -\nabla^*\nabla = P - I$

$$\begin{aligned} \langle f, Pf\rangle &= \langle f, (P-I)f\rangle + \langle f, f\rangle \\ &= -\langle \nabla f, \nabla f\rangle + \langle f, f\rangle \\ &||f||_2^2 - D_p(f) \\ &\leq (1-\kappa)||f||_2^2 \end{aligned}$$

P is self-adjoint, so

$$\langle Pf, Pf\rangle = \langle f, P^2 f\rangle \leq \left(1 - \frac{1}{x^2}\right)||f||_2^2$$

$S(P) = ||P|| \leq 1 - \frac{1}{x^2} < 1$.
$(c) \Rightarrow (b)$: P is self-adjoint $\Rightarrow ||P|| = \rho(P) < 1 \Rightarrow$

$$D_P(f) = ||f|_-2^2 - \langle f, Pf\rangle \geq (1 - \rho(P))||f||_2^2$$

$(c) \Leftrightarrow (d)$: $(I - P)Gf = f, \forall f \in l_0(X) \Rightarrow$ "$G = (I-P)^{-1}$". Finish $(d) \Rightarrow (c)$. □

Remark 0.116. Amenable groups are quasi-isometry invariant.

Lecture 22. October 30, 2009

Now we will be using *Random Walks on Groups and Random Transformations* by Alex Furman as a reference (available on his website).

1. Poisson Boundary: A topological approach

Let G be locally compact. For instance G discrete, Lie group, or a p-adic group.

Definition 1.1. A probability measure μ on G is **admissible** if the semi-group generated by the support of μ is G.

Definition 1.2. A **Markov operator** P is $(Pf)(g) = \int_G f(g,h)d\mu(h)$.

$$\mathcal{H}^\infty(G,\mu) = \{\text{bounded } \mu - \text{harmonic functions}\}$$

Definition 1.3. f is **μ-harmonic** if $f = Pf$.

Definition 1.4. $f : G \to \mathbb{R}$ is **left uniformly continuous** functions if $\forall \epsilon > 0, \exists$ a neighborhood U of $1 \in G$ such that $|f(gh) - f(h)| < \epsilon, \forall g \in U, \forall h \in G$.

$$B_{\text{luc}} = \{f : G \to \mathbb{R} \mid f \text{ is left uniformly continuous}\}$$

$\mathcal{H}^\infty_{\text{luc}}(G,\mu) = \{$bounded, left uniformly continuous μ-harmonic functions$\} \subset B_{\text{luc}}, \mathcal{H}^\infty(G,\mu)$

Remark 1.5. Any bounded uniformly continuous harmonic function is the pointwise limit of left uniformly continuous functions.

Proof. "Mullify" by integrating against a nice bump function. □

Lemma 1.6. *$\forall f \in H^\infty(G,\mu)$, the limit:*

$$\widetilde{f}(g,\omega) = \lim_{n\to\infty} f(g\omega_1\omega_2\cdots\omega_n)$$

exists $\forall g \in G$ and $\mathbb{P}$-almost everywhere sequence $\omega \in \prod_{i=0}^\infty G = G^\infty$.

Denote $\widetilde{f}(\omega) = \widetilde{f}(1,\omega)$.

Definition 1.7. A sequence of $\mathbb{R}$-valued functions $\{g_n\}$ is **martingale** if the g_n are measureable with respect to $\mathcal{F}_n$ and $E\{g_n \mid F_{n-1}\} = g_{n-1}$.

Note: Fix a measure μ. $E(f) =$ the expectation of $f = \int f d\mu$.

Theorem 1.8 (Martingale Theorem)**.** *If $\exists c$ such that $\forall n$, $E(\varphi_n) = \int \varphi_n d_\mu < c < \infty$, then $\varphi_n \to \varphi$ with probability 1, where φ is measurable with respect to $\mathcal{F}$.*

Proof. This involves the "Maringale Theorem." Supose that $\mathcal{F}$ is a σ-algebra and $\mathcal{F}_n$ is a sequence of nested σ-algebras ($\mathcal{F}_n \subset \mathcal{F}_{n+1}$). $\mathcal{F}$ is generated by the $\mathcal{F}_n$, i.e. $\mathcal{F}$ is the smallest σ-algebra which contains all $\mathcal{F}_n$. Fix a probability measure μ. Let $\{g_n\}$ be a martingale sequence. $E(\varphi_n \mid F_{n-1}\} = \varphi_{n-1}$.

$$\Omega = \{\text{sequence } (g_n)_{n=0,1,\ldots} \mid g_n \in G\}$$

$$\mathcal{F}_n = \sigma - \text{algebra } = \text{ products of } \sigma\text{-algebras of } G \text{ for the first } n \text{ factors}$$

Then $f_n = f(g\omega_1\cdots\omega_n)$ is a function on Ω measurable with respect to $\mathcal{F}_n$.

$$E(f_{n+1} \mid \mathcal{F}_n) = \int f(g\omega_1\cdots\omega_n h)d\mu(h) = f(g\omega_1\cdots\omega_n) = f_n$$

□

Lecture 23. November 2, 2009

Ergodicity
Let G be a group that acts on a probability space (X,μ) such that G preserves sets of measure zero with respect to μ (i.e. $A\subset X, \mu(A)=0 \Rightarrow \mu(gA)=0$. μ is a **quasi-invariant measure**. Can define a **Radon-Nikodym Derivative**;

$$g_*\mu=\rho(g,x)\mu,\ \text{i.e.}\ \int f dg_*(\mu)=\int f\rho(g,x)d\mu$$

where $f:X\to\mathbb{R}$, $g_*\mu$ is a pushforward measure, and $\rho: G\times X\to\mathbb{R}$.

Definition 1.9. μ iis **ergodic** if the only invariant measure subsets $A\subset X$ (i.e. $\forall g\in G, gA=A$) have measure zero or $X-A$ has measure zero.

Example 1.10. $\mathbb{Z}$ acts on S^1 by irrational rotations. Then λ =Leb on S^1 is ergodic.

Idea:

Lemma 1.11 (Idea of Lemma)**.** *ergodic* $\Leftrightarrow$ *every G-invariant $L^1(L^2)$-function is constant.*

Let R_α be a rotation by α and $\{R_\alpha^n\}=S^1$. For $f\in L^2(S^1)$ that is R_α-invariant $\Rightarrow f$ is R_α^n invariant $\Rightarrow f$ is S^1-invariant $\Rightarrow f$ is constant.

Example 1.12. $\Gamma=\pi_1(K)$, where K is a compact surface of genus ≥ 2. Γ acts on $S^1=\partial(\mathbb{H}^2)$. Claim: this action is ergodic.

Example 1.13. Let G be a locally compact group. $\Omega=G^{\mathbb{N}}=\{\text{sequence } (g_1,\dots,g_n,\dots), g_i\in G\}$, μ on G is an admissible measure. $\mathbb{P}$-measure on Ω. $\prod_{i=1}^\infty\mu$. **Bernoulli shift**:

$$\theta:\Omega\to\Omega \text{ shift map, so that } \theta(g_1,g_2,\dots,g_n,\dots)=(g_2,g_3,\dots)$$

Warning: θ is not invertible, but A is invariant if $\theta^{-1}(A)=A$. Claim: θ is ergodic.

Note: Classically consider $G=\mathbb{Z}_2=\{0,1\}$.

Theorem 1.14 (Ergodic Decomposition Theorem)**.** *Let X be a compact metric. G acts on X by homeomorphisms. Let μ be a quasi-invariant probability measure under G. Then we can decompose μ as:*

$$\mu=int_{\alpha\in(Y,\nu)}\mu_\alpha d\nu(\alpha)$$

such that the μ_α are ergodic G-quasi-invariant measures.

This comes from the Choquet Theorem:

Theorem 1.15. *$\mathcal{C}\subset B$ is a convex, w-compact set. Define $c\in\mathcal{C}$ is **extremal** if $c=\alpha c_1+(1-\alpha)c_2$, where $c_1,c_2\in\mathcal{C}$ and $\alpha\neq 0$. $\forall c\in\mathcal{C}, c=\int_{\alpha\in Y}c_\alpha d\nu(\alpha)$, where $Y=\{$extremal points of $\mathcal{C}\}$.*

Example 1.16. $T=\begin{pmatrix}2&1\\1&1\end{pmatrix}$ on $T^2=\mathbb{R}^2/\mathbb{Z}^2$ leaves λ-Lebesgue measure invariant. λ is ergodic with respect to T. For $A\in\mathrm{SL}(n,\mathbb{Z})$ and T^n, A is ergodic $\Leftrightarrow$ no eigenvalue of A is a root of unity.

Let G be locally compact and μ admissible. Recall:

$$f\in\mathcal{H}^\infty(G,\mu)=\{\text{l.u.c. bounded harmonic functions}\}\rightsquigarrow\widetilde{f}(g,\omega)=\lim_{h\to\infty}f(g\omega_1\omega_2\cdots\omega_n)$$

exists $\forall g$ and $\mathbb{P}$-almost everywhere ω.
Construct μ-harmonic functions:
Suppose G acts on (M,ν), where μ is a probability measure on M.
Problem: There may be no G-invariant probability measures.
New idea: Let μ be a measure on G. Then for $G\times M\xrightarrow{A}M$, $A_*(\mu\times\nu)\equiv\mu*\nu$.

Definition 1.17. Let (G,μ) and (M,ν). ν is **stationary** with respect to μ (or μ-stationary) if $\mu * \nu = \nu$. If ν is μ-stationary, then (M,ν) is a (G,μ)-space.

Lemma 1.18. *μ-stationary measures always exist.*

Proof. $* : \mathcal{P}(M) \xrightarrow{\nu} \mathcal{P}$, probability measure on M. $\mu * \nu, \mathcal{P}(M) \ni \nu \mapsto \mu * \nu = *(\nu)$. $\lim_N \frac{1}{N+1}\sum_{n=0}^{N} *^n(\nu)$. □

Lecture 24. November 4, 2009

R.N. Derivative:
$\mu = f \cdot \lambda$, where μ and λ are measures, then $\frac{d\mu}{d\lambda}(x) = f(x)$.
Special Case: $\mu = g_*(\lambda)$. Suppose μ and $\lambda$are absolutely continuous with respect to each other (have same null sets) $\Rightarrow \mu = \rho(g,x)\lambda$.
Super special case: g diffeomorphism of a manifold, λ a smooth volume form. $\rho(g,x) =$Jacobian of g.

Theorem 1.19 (Fuestenberg)**.** *Suppose (M,ν) is a measurable (G,μ)-space (ν is μ-stationary). Given $\phi \in L^\infty(M,\nu)$, define*

$$f_\phi(g) = \int_M \phi(gx)d\nu(x) = \int_M \phi dg(x).$$

Note that $\int_M \phi(gx)d\nu(x) = \int_M \phi(x)\rho(g,x)d\nu(x)$, where ρ is the Radon-Nikodym derivative. Then

- **(a):** $f_\phi \in \mathcal{H}^\infty(G,\mu)$
- **(b):** *f_ϕ is constant $\forall \phi \in L^\infty(M,\nu) \Leftrightarrow \nu$ is G-invariant.*

Furthermore, suppose (M,ν) is a compact G-space. Then:

- **(a'):** $f_\phi \in \mathcal{H}^\infty_{luc}(G,\mu)$
- **(b'):** *f_ϕ is constant $\forall \phi \in \mathcal{C}(M) \Leftrightarrow \nu$ is G-invariant.*

Of Theorem. f_ϕ is bounded by $||\phi||_\infty$.

$$\begin{aligned}\int_G f_\phi(gg')d\mu(g') &= \int_G\int_M \phi(gg'x)d\nu(x)d\mu(g')\\ &= \int_G\int_M \phi(g,y)d(\mu*\nu)(y)\\ &= f_\phi(g)\end{aligned}$$

where we get last equality from $\mu * \nu \leftrightarrow \nu$. Therefore f_ϕ is harmonic. The second equality follows from:

$$\int_M f(z)d(\mu*\nu)(z) = \int_G\int_M f\circ\alpha(g',z)d\mu(g')d\nu(z) = \int_G\int_M f(g'z)d\mu(g')d\nu(z)$$

replacing y by $g'(x)$. This proves (a). □

Lemma 1.20. *Suppose (M,ν) is a compact G-space. For $\mathbb{P}$-a.e. $\omega = (\omega_1,\omega_2,\ldots,\omega_n,\ldots) \in \Omega = G^{\mathbb{N}}$ of the right μ random walk, then $\exists$ a limit measure ν_ω on M such that $\nu_\omega = \lim_{n\to\infty}\omega_1\omega_2\cdots\omega_n\nu$.*

Proof. $\frac{P}{\nu_\omega(\phi)}, \phi \rightsquigarrow f_\phi : G \to \mathbb{R}$, $h : G \to \mathbb{R}, \widetilde{h} = \lim_{n\to\infty} h(\omega_1\cdots\omega_n)$ is defined for $\mathbb{P}$-a.e. ω. Set $\nu_\omega(\phi) = \widetilde{f}_\phi(\omega)$. □

We expect that ν_ω will be a point measure (this occurs $\mathbb{P}$-a.e).

Definition 1.21. A compact (G,μ)-space (M,ν) is a **compact** (G,μ)**-boundary** if ν_ω is a Dirac measure for $\mathbb{P}$-a.e. $\omega \in \Omega$.

We get a map $\partial : (\Omega, \mathbb{P}) \to (M, \nu)$ given by $\omega \mapsto$ the support of ν_ω.
Sometimes one wants a meas. definition

Definition 1.22. An abstract (G,μ)-space (M,ν) is a (G,μ)**-boundary** if the transform $\phi \mapsto f_\phi$ satisfies $\widetilde{f}_{\phi\psi} = \widetilde{f}_\phi \widetilde{f}_\psi, \forall \phi, \psi \in L^\infty(M,\nu)$ and $\mathbb{P}$-a.e. ω.

Notes:

(1) Let $M = S^1$ with Lebesgue measure ν. $\phi \in L^\infty(S^1,\nu) \rightsquigarrow f_\phi$ harmonic (from the Fustenberg theorem) and $\phi \rightsquigarrow$ (via Poisson integral) a harmonic function of the disk.

(2) Suppose (M,ν) is a boundary. Then we get a map $L^\infty(M,\nu) \to \mathcal{H}^\infty(G,\mu)$ by $\phi \mapsto f_\phi$. Respectively, we get a map: $\mathcal{C}(M) \to \mathcal{H}^\infty_{\text{luc}}(G,\mu)$, where $(C)(M) =$ continuous functions on M. These are both isometric embeddings.
For show that these are isometric, use the following. $\phi(p), \delta_p = \nu_\omega$ same ω and $\lim_{n\to\infty} f_\phi(\omega_1 \cdots \omega_n) = \widetilde{f}_\phi(\omega) = \phi(p)$.

Lecture 25. November 6, 2009

A fun aside: superquick proof that $\Gamma = \pi_1(\text{compact surface}) = \pi_1(\mathbb{H}^n/\Gamma)$, where Γ is an n-dimensional hyperspace. Γ acts S^1, Lebesgue. This action is ergodic more generally same for $\mathbb{H}^n/\Gamma$. Γ acts on S^{n-1} Lebesgue. *Proof:* Suppose Γ is not ergodic on S^1. Let f be a Γ-invariant function. f invariant function $\rightsquigarrow H^\infty(\mathbb{H}^2, \mathbb{R}\text{-metric})$ via a Poisson integral, so $f \mapsto F$ which is Γ invariant and $F^* : \mathbb{H}^2/\Gamma \to \mathbb{R}$ is harmonic. We can bring F^* back up to a Γ-invariant function, so it must be constant (use Fustenberg).

Construction of the Poisson Boundary: *(Fustenberg)*
Recall: (X,ν) boundary $\rightsquigarrow (X^*,\nu^*)$ is a boundary.

$$L^\infty(X^*,\nu^*) \hookrightarrow L^\infty(X,\nu) \xhookrightarrow{\text{isometry}} H^\infty(G,\mu)$$

Definition 1.23. (W,ν) is a **Poisson boundary** if it is a maximal (G,μ)-boundary.

Idea for construction: Look for a maximal suitable subspace of $H^\infty(G,\mu)$.

$$\mathcal{A} = \{f \in B_{\text{luc}}(G) \mid \forall g \in G \text{ and for } \mathbb{P}-\text{a.e. } \Omega \longleftarrow G^{\mathbb{N}}, \exists \widetilde{f}(g,\omega) = \lim_{n\to\infty} f(g\omega_1\omega_2\cdots\omega_n)\}$$

$\mathcal{A}$ with $+, \cdot$ is a commutative Banach algebra (in fact C^*-algebra).

$$Z = \{f \in \mathcal{A} \mid \overline{f}(g,\omega) = 0, \forall g \in G, \text{ a.e. } \omega \in \Omega\}$$

$\mathcal{A}/Z$ is a commutative Banach algebra.

Theorem 1.24 (Gelfand,Naimark)**.** *A C^*-algebra, $B = \mathcal{C}(X) = \{$continuous functions on $X\}$, where X is a compact space.*

Idea:

$$X = \{\text{maximal ideals in } B\} = \{\alpha : B \to \mathbb{R}, \alpha \text{ respects } +, \cdot\}$$

So $\alpha(b_1b_2) = \alpha(b_1)\alpha(b_2), \alpha(1) = 1$. Endow $X$with the weak topology.
$\overline{B} =$ Gelfand dual of $\mathcal{A}/\mathbb{Z}$ and $\mathcal{A}/\mathbb{Z} = \mathcal{C}(\overline{B})$. Get an action of G on $\overline{B}$. $(G,\mu) \rightsquigarrow (\overline{B}, ?)$: for $\mathcal{A} \ni f \xrightarrow{L_g} \int_\Omega \overline{f}(g,\omega) d\mathbb{P}(\omega)$. $L_g|_Z = 0 \rightsquigarrow L_g$ induces a functional on $\mathcal{A}/Z$, i.e. a measure ν_g on $\overline{B}$. Check:

(1) $\nu_g = g_*\nu_1$
(2) ν_1 is μ-stationary
(3) $(\overline{B}, \nu_1)$ is a (G,μ)-boundary

Claim: $\mathcal{A} = Z \oplus \mathcal{H}^\infty(G,\mu)$ (i.e. $A/Z = \mathcal{H}^\infty(G,\mu)$). Then $L^\infty(\overline{B},\nu) \cong \mathcal{H}^\infty(G,\mu)$.

Proof of Claim. Check transversality of Z and $\mathcal{H}^\infty(G,\mu)$. Suppose $f \in Z \cap \mathcal{H}^\infty(G,\mu)$. Then $f(g) = \int_\Omega \overline{f}(g,\omega)d\mathbb{P}(\omega) = 0$ since $f \in Z$ and $f \in \mathcal{H}^\infty(G,\mu)$ by assumption. Therefore $Z \cap \mathcal{H}^\infty(G,\mu) = \{0\}$.
Check $\mathcal{A} = Z + \mathcal{H}^\infty(G,\mu)$. Let $h \in \mathcal{A}$ and denote $\phi \in \mathcal{C}(\overline{B})$ which correpsonds to $h + Z$. Let $f_\phi \in \mathcal{H}^\infty(G,\mu)$ = harmonic functions we get by F' berg. Then $\overline{f}_\phi(\omega) = \lim_{n\to 0} f_\phi(\omega_1\omega_2\cdots\omega_n) \Rightarrow h - f_\phi \in Z$. □

Why is $\overline{B}$ maximal? Let B' be another boundary.

$$\mathcal{C}(B') \longrightarrow \mathcal{H}^\infty_{\text{luc}}(G) \longrightarrow \mathcal{C}(\overline{B} \text{ and } B' \longleftarrow \overline{B}$$

Problem: Can we identify Poisson boundaries in particular situations?

- **(A):** G is a Lie group and μ an admissable measure
- **(B):** M is a compact Riemannian manifold with negative sectional curvature, $\Gamma = \pi_1(M)$. Γ Gromov hyperbolic.
- **(C):** Moduli group
- **(D):** Γ amenable group ?

$\Gamma = \pi_1$(compact surface)
Exhibit A: (S^1,Leb.) $= P^1\partial$.

$$\text{PSL}(2,\mathbb{R}) = \text{Isom}\mathbb{H}^2 = \text{SL}(2,\mathbb{R})/\{\pm I\}$$

SL$(2,\mathbb{R})$, $z \mapsto \frac{az+b}{cz+d}$ so $0 \mapsto \frac{b}{d} = 0 \Rightarrow b = 0$. So $P = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$ and $S^1 = \text{SL}(2,\mathbb{R})/P$.

Lecture 26. November 9, 2009

Remark 1.25. Let G be a group, (G,μ) admissible and G acts on X. Stationary probability measures ν such that $\mu * \nu = \nu$:

1. stationary measures always exist
2. $\{\nu \mid \mu \text{ stationary }\}$ is convex and weakly-closed, thus has extreme points
3. correspondence to shift invariant measure

$C \subset B^*$ is weakly-closed and convex.

$$\text{Ex}(C) = \text{extreme points of } C = \{c \in C \mid c \text{ is not a convex combination of points in } C\}$$

For $c \in C$, $c = \int_{e\in\text{Ex}(C)} e d\lambda(e)$ (ergodic decomposition)

Example 1.26. G-action on X, $C = \{G\text{-invariant probability measure}\}$, Ex$(C)$ = ergodic invariant probability measure.

$(\Omega,\mathbb{P}) =)G^{\mathbb{N}}, \mu^{\mathbb{N}}), \theta : \Omega \to \Omega, (\theta\omega)_i = \omega_{i+1}$. X-G space,

$$T; \Omega \times X \to \Omega \times X \text{ is given by } T(\omega, x) = (\theta\omega, \omega_1 x)$$

Proposition 1.27.

- **(a):** *$\mathbb{P} \times \nu$ is T-invariant $\Leftrightarrow$ ν is μ-stationary*
- **(b):** *$\mathbb{P} \times \nu$ is T-ergodic $\Leftrightarrow$ ν is an extremal μ-stationary measure*

Proof. Easy. Just calculate. □

Theorem 1.28 (Fustenberg)**.** *Let M be a compact metric G-space and μ an admissible measure on G. Then the G-action on the space $\mathcal{P}(M) = \{$probability measures on $M\}$ contains a (G,μ) boundary.*

Why bother?

Corollary 1.29. *Let B be the Poisson boundary of G. Then $\exists$ a G-equivariant map $B \to \mathcal{P}(M)$.*

Proof. By universailty of B and the theorem. $\square$

More (but vague) motivation: The Poisson boundary of $\mathrm{SL}(n,\mathbb{R})$ is $B = \mathrm{SL}(n.\mathbb{R}) / \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} =$ the flag variety. In fact, B is also a Poisson boundary for $\mathrm{SL}(n,\mathbb{Z})$.
$\Gamma = \mathrm{SL}(n,\mathbb{Z}) \xrightarrow{r} ho\mathrm{SL}(M,\mathbb{R})$, thus Γ acts on $\mathrm{SL}(M,\mathbb{R}) / \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.

Theorem 1.30 (Super-rigidity, Margulis)**.** *For $n \geq 3$,*

$$\begin{array}{ccc} SL(n,\mathbb{R}) & & \\ \uparrow \subset & \searrow & \\ \Gamma = SL(n,\mathbb{Z}) & \xrightarrow{\rho} & SL(M,\mathbb{R}) \end{array}$$

If $\overline{\rho(\Gamma)}$ is not compact.

Proof of Fustenberg Theorem. Let $V = \mathcal{P}(M)$ and $V \hookrightarrow \mathcal{P}(V)$ given by $v \mapsto \delta_v$. We also have a map bary: $\mathcal{P}(V) \to V$ barycenter map given by extending continuously the map $\sum_i p_i \delta_{v_i} \mapsto \sum_i p_i v_i$ to $\mathcal{P}(V)$. In particular, $\mathcal{P}(V) \ni \alpha = \int_{v \in V} d\beta(\delta_v) \mapsto \int_{v \in V} d\beta(v)$, for some probability measure β. Let ν be a μ-stationary measure on M.

Lemma 1.31. *There is a decomposition: $\nu = \int_{\omega \in \Omega} \nu_\omega d\mathbb{P}(\omega)$.*

Proof. Recall that $\nu_\omega(\phi) = \widetilde{f_\phi} = \lim_{n\to\infty} f_\phi(\omega_1 \cdots \omega_n)$.

$$\begin{aligned} \int_{\omega \in \Omega} \nu_\omega d\mathbb{P}(\omega) &= \int_{\omega \in \Omega} \lim_{n\to\infty} f_\phi(\omega_1 \cdots \omega_n) d\mathbb{P}(\omega) \\ &= \lim_{n\to\infty} \int_{\omega\in\Omega} \int_{x \in M} \phi(\omega_1 \cdots \omega_n x) d\nu(x) d\mathbb{P}(\omega) \\ &= \lim_{n\to\infty} \int_{\omega\in\Omega} \int_{x \in M} \phi(\omega_1 \cdots \omega_n x) d\nu(x) d\mu(\omega_1) \cdots d\mu(\omega_n) \\ &= \lim_{n\to\infty} \int_M \phi(x) d\nu(x) \end{aligned}$$

Using boundedness and Fatou, we can bring the limit outside of the integral. The last equality follows since ν is μ-stationary. $\square$

Assume the lemma is true. Define $\widetilde{\nu} \in \mathcal{P}(V)$ as:

$$\widetilde{v} = \int_{\omega \in \Omega} \delta_{\nu_\omega} d\mathbb{P}$$

$\mathrm{bary}(\widetilde{\nu}) = \nu$ and $\widetilde{\nu}$ is μ-stationary on V. For $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\mathrm{bary}(\omega_1 \cdots \omega_n \widetilde{v}) = \omega_1 \cdots \omega_n \mathrm{bary}(\widetilde{\nu}) = \omega_1 \cdots \omega_n \nu \xrightarrow[n\to\infty]{} \nu_\omega$$

So $\omega_1 \cdots \omega_n \widetilde{v} \to \delta_{\nu_\omega}$. Hence $\mathrm{Supp}\widetilde{v} \subset V = \mathcal{P}(M)$ is a G-μ-boundary. $\square$

Lecture 27. November 11, 2009

Main Problem: How to determine the Poisson boundary
References: Fustenberg's paper Annals 1963, Glasner's *Proximal Flows* (free online on Springer)
Boundary (G,μ)-space (M,ν). Characteristic feature: for $\mathbb{P}$-almost every $\omega \in \Omega, \omega_1 \cdots \omega_n \nu \to \delta_m$, some $m \in M$.

Topological Category:
Let G be a group and X a G-space (i.e. G acts on X by homeomorphisms).

Definition 1.32. X is **proximal** if $\forall x, y \in X, \exists$ a sequence $(g_n) \in G$ such that $g_n x, g_n y$ are arbitrarily close (i.e. $\lim_{n\to\infty} g_n x = \lim_{n\to\infty} g_n y$).

Definition 1.33. X is **strongly proximal** if the action of G on the space of probability measures $\mathbb{P}(X)$ is proximal (i.e. given two probability measures $\nu_1, \nu_2, \exists g_n$ such that $g_n \nu_i \to \nu$).

Examples:

(1): Isometric actions do not lead to proximal spaces.

(2): $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ on T^2 has periodic points so it is not proximal.

(3): $\mathrm{PSL}(2, \mathbb{R})$ on S^1 is proximal and strongly proximal.

(3'): $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ on S^1, $\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$ is almost proximal but it has two fixed points so it is not proximal.

(3"): $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ on S^1 (viewing S^1 as $\mathbb{R} \cup \{\infty\}$) is strongly proximal.

(3"'): $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ on S^1 is proximal and strongly proximal, but not minimal since ∞ is a fixed point.

Definition 1.34. G acts on a topological space X is **minimal** if all G-orbits are dense.

Proximal G-space X, Y a G-space and $\pi : X \to Y$ with $\pi(gx) = g\pi(x)$, then Y is proximal.

Theorem 1.35 (Abstract Nonsense 1)**.** *There exists a minimal proximal G-space.*

Proof. See Glasner. ☐

Theorem 1.36 (Abstract Nonsense 2)**.** *There exists a minimal strongly proximal G-space.*

Call the minimial strongly proximal G space: $\pi_S(G)$ or B.-Fustenberg boundary.
Connection to Poisson Boundaries:
(G, μ) with μ admissible and (B, ν) a Poisson boundary.
Claim: There is always a map $(B, \nu) \to \pi_S(G)$.
$\pi_S(G)$ is nice in part because it does not depend on μ.

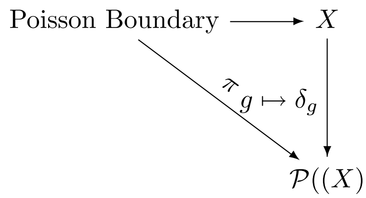
Remark 1.37. G ammenable $\Rightarrow \exists G$-invariant probability measure ν on $\pi_S(G) = X$. Pick $p \in X$. There exists $g_n \in G$ such that $\mu = (g_{n*})\mu \to \lambda$ and $(g_{n*})\delta_p = \delta_{g_n p} \to \lambda$. For some $q \in X, \mu = \delta_q$. So $\pi_S(G)$ =point.

Lecture 28. November 13, 2009

Theorem 1.38. *Suppose (G, μ) with μ admissible and X a minimal strongly proximal G-space, then there is a map Poisson Boundary $\xrightarrow{\pi}$ X.*

Idea of Proof. Poisson boundary $\xrightarrow{\pi} \mathcal{P}((X)$. Let ν be a measure on the poisson boundary.
Claim: $\pi_*(\nu)$ is a Dirac ∂
$\exists g_n \in G$ such that $g_n(\pi_* \nu) \to \delta_p, p \in X$.
Claim: π_*(Poisson boundary) is minimal
Poisson boundary compact space. π_*(Poisson boundary) is compact $\Rightarrow \pi_*$(Poisson boundary($\cap$

Dirac measure $\neq \emptyset$. Show that every point in the Poisson boundary maps to a Dirac measure.



□

Fustenberg Boundary of SL$(n, \mathbb{R})$:

(1) G amenable $\Rightarrow \pi_S * G(= \{\text{point}\}$
(2) $G \supset S$ cocompact space, G/S compact

Example 1.39 (Example of (2)). Let $G =$SL$(n, \mathbb{R})$ and $S = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.
G acts on $\mathbb{R}^n$. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$.

$$\mathbb{R}e_1 \subset \mathbb{R}e_1 \oplus \mathbb{R}e_2 \subset \cdots \subset \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \cdots \oplus \mathbb{R}e_{n-1} \subset \mathbb{R}^n$$

is a **full flag** (increasing sequence of subspaces in $\mathbb{R}^n$).

Examples:

(a): Flag with only one subspace, only dimension $k \rightsquigarrow G_{x_k} = \{k$-dimensional subspace of $\mathbb{R}^n\}$
(b): $G_{x_1} = \mathbb{RP}^{n-1}$
(c): SL$(2, \mathbb{R})/S = \mathbb{RP}^1 = S^1$

Given 2 full flags: $0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq \mathbb{R}^n$ and $0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_{n-1} \subsetneq \mathbb{R}^n$, $\exists g \in$SL$(n, \mathbb{R})$ such that $gV_i = W_i$.

Corollary 1.40. *$G/S = \{$full flags in $\mathbb{R}^n\} = \mathcal{F}$.*

Theorem 1.41. *The Fustenberg boundary for SL$(n, \mathbb{R}) = G/S = \{$full flags$\}$.*

Proposition 1.42. *Let G (be general) and $S \subset G$ cocompact. If X is a strongly proximal G-space, then the restriction of the actiion to S is still strongly proximal.*

$\mu_i \rightsquigarrow g_n\mu_i, i \in \{1, 2\}$. Write $g_n = k_n s_n$, where we pick $k_n \subset C$ and $C \subset G$ is compact. We can do this since G/S is compact. Then $s_n\mu_i \xrightarrow{\nu} k_n^{-1}\nu \mapsto \overline{\nu}$.

Lemma 1.43. *If X is a minimal strongly proximal G-space, then $X \hookrightarrow P(X)$ given by $p \mapsto \delta_p$ and $X \subset \mathcal{P}((X)$ is the unique minimal set in $\mathcal{P}(X)$.*

$\pi_S(G)$, S fixes a probability measure μ on $\pi_S(G)$ and S is strongly proximal on $\pi_S(G) \Rightarrow \exists s_n \in S$ such that $\mu = (s_n)_*\mu \to$ Dirac measure $\Rightarrow \mu =$Dirac$= \delta_p$ for some $p \Rightarrow S$ fixes aa point on $\pi_S(G)$.
$p, q \in \pi_S(G) \Rightarrow \exists g_n \in G$ such that $g_n p \to q$. As before, $g_n = k_n s_n$ where $k \in C$ compact and $s_n \in S$ with $k_n \to k \in G$. Then $g_n p = k_n s_n p = k_n p \to kp$ and $g_n p \to q$ so $q = kp$. Therefore $\pi_S(G) = G/H, H$ closed and $G \supset S$. *Claim:* $H = S$. $G/S \to G/H$.

Lemma 1.44. *G acts strongly proximally on G/S.*

Lecture 29. November 20, 2009

Theorem 1.45. *Let f be a bounded continuous function on G/K and $\widetilde{f}$ its lift to G (i.e. $G \xrightarrow{\widetilde{f}} G/K \xrightarrow{f} \mathbb{R}$). TFAE:*

a: *$\exists \phi \in L^\infty$(Furstenberg boundary, m) such that $\widetilde{f}(g) = \int_B \phi(gx)dm(x)$*

b: *$\widetilde{f}$ is μ harmonic for every spherical measure μ on G (i.e. μ is K bi-invariant)*
c: $\widetilde{f}(g) = \int_K \widetilde{f}(gkg')dk, \forall g, g'$

Furstenberg's most crucial theorem:

Theorem 1.46. *μ absolutely continuous measure on G. $f(g)$ is a bounded μ-harmonic function on G such that $f(gk) = f(g), \forall k \in K, g \in G$. Then f is constant.*

Need to average on the other side of where you mod out by K. Note that f is μ-harmonic means $f(g) = \int_G f(g'g)d\mu(g')$. Really f here is a lift of a function of $D \equiv G/K$. Reformulation of the same theorem:

Theorem 1.47. *$f : D \to \mathbb{R}$ is a bounded μ-harmonic, μ absolutely continuous meaure on G (i.e. $f(p) = \int f(g'p)d\mu(g')$). Then f is constant.*

Corollary 1.48. *If $g : G \to \mathbb{R}$ is a μ-harmonic function for some spherical measure the it is right K-invariant.*

Corollary 1.49. *If f is a continuous bounded μ-harmonic function for some spherical measure μ, then f is harmonic.*

Proof. Fix $g \in G$. Set $F(g') = \iint_K \widetilde{f}(g'kg)dk$. Then $F(g')$ is μ-harmonic.

$$\begin{aligned} \int_G F(g''g')d\mu(g'') &= \int_G \int_K \widetilde{f}(g''g'kg)dkd\mu(g'') \\ &= \int_K \int_G \widetilde{f}(g''g'kg)d\mu(g'')dk \\ &= \int_K \int_G \widetilde{f}(g'kg)dk \\ &= F(g') \end{aligned}$$

Therefore F is constant (by Furstenberg). Let $g = 1$. $F(g) = \int_K \widetilde{f}(g'k)dk = \widetilde{f}(g')$ by K-invariance. Then $\widetilde{f}$ is constant since F is constant. □

Corollary 1.50. *Let μ be an absolutely continuous measure on G. Let $(Poiss, \nu)$ be its Poisson boundary. Then K acts transitively on Poiss.*

Since Poiss $= G/H, H \subset P \to G/P =$Furstenberg boundary.

Proof. Suppose K is not transitive on Poiss. Then $\exists K$-invariant closed, disjoint subsets $P_1, P_2 \subset$ Poiss. Use crucially that K is compact (for instance use Poiss$/K$ is Hausdorff and two disjoint points in it lead to two different orbits). Let ϕ be a continuous function on Poiss such that $\phi_{P_1} = 1$ and $\phi_{P_2} = 0$. Consider $\psi(x) \equiv \int_K \phi(kx)dk$. ψ is continuous, K-invariant, and non-constant on Poiss. Set $f(g) \equiv \int_{\text{Poiss}} \psi(gx)d\nu(x)$. f is μ-harmonic. Then $f(kg) = \in \psi(kgx)d\nu(x) \Rightarrow f(g) = \int \psi(gx)d\nu(x)$ (note this is opposite from the 2nd theorem in this lecture). $f \neq$ constant, which contradicts Furstenburg. □

Corollary 1.51. *If μ is absolutely continuous, then the Poisson boundary for μ is a finite cover of the Furtstenburg boundary.*

Corollary 1.52. *Let μ is absolutey continuous and $\mu^n \equiv \mu * \cdots * \mu$. Suppose $supp(\mu^n) \supset$ neighborhood 1. Then Poisson boundary for $\mu = G.P =$Furstenburg.*

A little measure theory:
Let ν_1, ν_2 be two measures and $\nu = \nu_1 + \nu_2$. Then ν is a measure and ν_1, ν_2 are absolutely continuous with respect to their sum ν. Define $\nu_1 \wedge \nu_2$ by $\frac{d(\nu_1 \wedge d\nu_2)}{d\nu} = \min\left(\frac{d\nu_1}{d\nu}, \frac{d\nu_2}{d\nu}\right)$.

Step 1:

Lemma 1.53. *Given* $c > 0, \exists \epsilon > 0$ *and* $n \in \mathbb{N}$ *such that if* $p_1, p_2 \in D$ *with* $d(p_1, p_2) < c$, *then* $(\mu^n * \delta_{p_1}) \wedge (\mu^n * \delta p_2))(D) > \epsilon$.