Symplectic 4–manifolds with $\kappa = 0$

Stefano Vidussi (joint w. Stefan Friedl)

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Canonical examples: Kähler surfaces, highly “non–generic”.
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If $M$ is symplectic, it admits an almost complex structure $J \in \text{End}(TM)$.

\textbf{Definition:} $\kappa := c_1(J) \in H^2(M, \mathbb{Z})$. 
Symplectic 4–manifolds with $\kappa = 0$

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Question: Are these the only examples?
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It is possible to construct several manifolds, using 2., whose Seiberg–Witten invariants are compatible with the condition $\kappa = 0$.
Main result

Our main result is that, when it comes to 2.2., none of these examples is new:

\[\text{Theorem: If } p: M \to N \text{ is symplectic, } \kappa = 0, \text{ then } M \text{ is a } T^2\text{–bundle over } T^2.\]

(Actually covers all symplectic manifolds \(M\) with \(\text{Kod}(M) = 0\).)

The theorem above is related with (and partially answers to) Conjecture: If \(M \to N\) is symplectic, then \(N\) fibers over the circle with a fiber \(\Sigma\) satisfying \(<e(M), [\Sigma]> = 0\).

Well–known (McCarthy) (using geometrization): \(N\) prime.

To simplify presentation: \(b_1(N) > 1\).

Two cases to consider: \(e(M) \in H^2(N, \mathbb{Z})\) torsion or not torsion.
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Two cases to consider: $e(M) \in H^2(N, \mathbb{Z})$ torsion or not torsion.
The case where $e(M)$ is torsion

**Lemma:** Let $M = S^1 \times N$; $M$ admits a symplectic form with $\kappa = 0 \iff N$ has vanishing Thurston norm.
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Proof: \(\implies\) wlog, we can assume that $[\omega] \in H^2(M, \mathbb{Z})$ and $H := PD[\omega]$ is represented by a symplectic surface (Donaldson), hence

$$\chi_-(H) = H \cdot H + \kappa \cdot H = H \cdot H.$$
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Write $\phi = p_*[\omega] \in H^1(N, \mathbb{Z})$: by Kronheimer’s refined adjunction,

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hence $\|\phi\|_T = 0$; wiggle $\omega$ to get vanishing Thurston norm on $N$. 

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$\Leftarrow$ $SW_{S^1 \times N} \ "=\ " \Delta_N$ and $\kappa \in \text{supp } SW_{S^1 \times N}$, hence

$$0 \leq \kappa \cdot \phi \leq \|\phi\|_A \leq \|\phi\|_T = 0 \implies \kappa = 0.$$
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**Theorem:** If $S^1 \times N$ is symplectic and $N$ has vanishing Thurston norm, then $N$ is a $T^2$–bundle over $S^1$, hence $S^1 \times N$ is a $T^2$–bundle over $T^2$.
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Let \( \alpha : \pi \to G \) be an epimorphism onto a finite group, \( N_G \stackrel{G}{\to} N \) the regular \( G \)--cover; if \( S^1 \times N \) is symplectic, \( \kappa = 0 \), then \( S^1 \times N_G \) is symplectic, \( \kappa_G = 0 \).
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The condition that $\kappa_G = 0$, together with Taubes’ “more constraints” implies that $SW_{S^1 \times N_G} = 1$, hence the twisted Alexander polynomial $\Delta^\alpha_N$ associated to the representation $\alpha : \pi \to G$ satisfies $\Delta^\alpha_N = 1$. 
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It follows: $\forall \alpha : \pi \to G$, $\Delta^\alpha_{N,\phi} = \text{ord}_{\mathbb{Z}[t^{\pm 1}]} H_1(\pi; \mathbb{Z}[G][t^{\pm 1}]) \neq 0$. 

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We have a Mayer-Vietoris type sequence for HNN extensions

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H_1(\pi; \mathbb{Z}[G][t^{\pm 1}]) \rightarrow H_0(A; \mathbb{Z}[G]) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}] \rightarrow H_0(B; \mathbb{Z}[G]) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}] \rightarrow H_0(\pi; \mathbb{Z}[G][t^{\pm 1}])
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The condition that $\kappa_G = 0$, together with Taubes’ “more constraints” implies that $SW_{S^1 \times N_G} = 1$, hence the twisted Alexander polynomial $\Delta_{N}^\alpha$ associated to the representation $\alpha : \pi \to G$ satisfies $\Delta_{N}^\alpha = 1$.

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But $H_i(\pi; \mathbb{Z}[G][t^{\pm 1}])$ are $\mathbb{Z}[t^{\pm 1}]$–torsion, hence

$$\text{rk}_{\mathbb{Z}} H_0(A; \mathbb{Z}[G]) = \text{rk}_{\mathbb{Z}} H_0(B; \mathbb{Z}[G]) \iff |\text{Im}(A \to G)| = |\text{Im}(B \to G)|.$$
Now as $\Sigma = T^2$, $A \subset \pi$ is abelian, hence separable: if by contradiction $A \subsetneq B$, then there exist an epimorphism $\alpha : \pi \to G$ s.t. $|\text{Im}(A \to G)| < |\text{Im}(B \to G)|$. 

Corollary: By going to a finite cover, we can easily obtain same result for nonzero torsion $e(M)$. 

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Solution: use more algebra & topology!

Lemma: If $\kappa = 0$ then $v_b(N, F) \leq 3$.

Proof: let $M \to S^1$ be obvious $S^1$–bundle over $N$. As for all $\alpha: \pi \to G$, $\kappa_G \in H^2(M_G, \mathbb{Z})$ is the only basic class, $\text{aug} \Delta N = \text{aug} SW M_G = 1$.

But if $v_b(N, F) > 3$, $\exists \alpha: \pi \to G$ s.t. $\text{aug} \Delta N = 0(p)$ (Turaev).
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\textbf{Lemma}: If $\kappa = 0$ then $\nu b_1(N, \mathbb{F}_p) \leq 3$. 
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**Theorem:** If $p : M \xrightarrow{S^1} N$ is symplectic, $\kappa = 0$ then $M$ is a $T^2$–bundle over $T^2$. 

**Proof:** If $N$ is a $T^2$–bundle over $S^1$, as $b_1(N) > 1$ it is also an $S^1$–bundle over $T^2$, hence the statement follows. Otherwise it satisfies one of the following:

1. $N$ has a nontrivial JSJ decomposition;
2. $N$ is Seifert-fibered with an incompressible $T^2$ that is not a fiber;
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If 3. holds, $\pi$ is f.g. linear group: by Lubotzky alternative, f.g. linear groups must be virtually solvable or $vb_1(N, \mathbb{F}_p) = \infty$; the first implies $N$ covered by torus bundle, impossible.
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1. It is known that if $M$ symplectic with $\kappa = 0$, $\Rightarrow vb_1(M) \leq 4$ (T.J.Li, Bauer); not known if $vb_1(M, \mathbb{F}_p) \leq 4$;

2. Lubotzky alternative holds, but we don’t have JSJ; are linear groups “interesting enough”;

3. If $M$ is hyperbolic, it has been conjectured that all SW invariants vanish.
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