1 Introduction

These notes are derived from the "Photons, Shmotons" thread that ran on sci.physics.research in the summer of 1997. (Later corrected to "Photons, Schmotons".) As editor, I've wielded a pretty free hand, correcting minor errors, imposing a consistent set of conventions, rearranging the order and condensing the discussion in places. On the other hand, I've kept some of the false starts and more rough-hewn derivations, even when slicker approaches appear later. I've omitted material that distracted from the main thrust of the argument (IMHO). Some of the omitted material has a natural place in a sequel; watch those photons on s.p.r!

—*Michael Weiss, editor*

These notes are also available in HTML form: see `http://math.ucr.edu/home/baez/photon/schmoton.htm`. The HTML version just has worse typesetting.

There is also a second group of notes, available at the same website, but currently only in postscript.

[Note: sections usually correspond to posts; the original author is given in the title, but I've chosen the section titles. —*ed.*]

2 John Baez: Things that Really Really Exist

Michael Weiss wrote:

> By the way, a mighty battle has been raging over on sci.physics over whether Photons Really Exist.

I'll join in as soon as I figure out the difference beween things that Exist and things that Really Exist — not to mention those that Really Really Exist.

> Throwing out 98.462% of this, I realize that I have but the vaguest conception of the sense in which Photons Really Exist according to QED. They can't be localized (if I understand Haag's comments about the Newton-Wigner wavefunction), they don't have identity — you can't say the photon absorbed by Nick Nitrogen over here is The Same Photon as was emitted a nano ago by Harry Hydrogen over there (do I have that right?) — a fine excuse for a particle!

If photons seem a bit dodgy, it's because:

1) They are bosons, so they are gregarious and like to travel in packs where you can't tell which one is to blame for anything.

2) They are massless, so it's hard to tell where the hell they are, more so than for your average particle.

Remember how a while back I showed how you could rewrite the Klein-Gordon equation as a first-order differential equation, which boiled down to Schrödinger's equation in the nonrelativistic limit? You need to pull this trick to get a nice "position operator" for a particle in relativistic quantum mechanics, analogous to the good old position operator for nonrelativistic quantum mechanics. It's called "Newton-Wigner localization". It doesn't work well for massless particles — which shouldn't be utterly surprising, since is no such thing as a nonrelativistic massless particle.

3) They carry no conserved charge, so it's easy to create or destroy a bunch of the rascals with the flick of a switch.

How unlike you and I, who Really Really Exist — at least according to ourselves! We are made of charged massive fermions. The Pauli exclusion principle makes us individualistic: we can't both be in the same place at the same time. Our massiveness makes us easier to localize, and we don't have to be running around at the speed of light all day. Finally, conservation of baryon and lepton number makes it hard to create or destroy us — or at least our underlying atoms. We naturally look with disdain on massless uncharged bosons.

3 Michael Weiss: A Trivial Exercise

Here's what ought to be a trivial exercise. Say we have a mono-chromatic beam of light, photon density N photons per square meter per second, angular frequency ω. Well, this is also a plane electromagnetic wave with amplitude A and angular frequency ω. Ought to be possible just using dimensional analysis to figure out the relationship of A to N and vice versa.

Last time I tried that, I didn't get what I expected…

4 Cinematic Interlude

Camera pans in on John Baez, strapped in a chair. A maniacal Michael Weiss hovers over him, holding a dentist's drill, whose tip contains, instead a diamond, a gleaming photon. Michael asks with a grating voice that sends phonons racing up and down one's spine (simultaneously):

> "Is it real?"

Baez groggily looks around. "Huh. Most people just post to sci.physics.research and hope for someone to answer, not kidnap the moderator and strap him to a dentist's chair! I know I'm overdue for a checkup, but this is ridiculous. Are you getting back at me for avoiding questions about ontology, or something? Is *what* real, anyway?"

"Shaddup, wiseguy." Weiss clobbers Baez with a cosh.

After a few more bizarre special effects, changes of scene, and a whole lot of dreamy music, Baez wakes up. All he remembers is a question...

5 John Baez: A Rather Odd Question

> Say we have a mono-chromatic beam of light, photon density N photons per square meter per second, angular frequency ω. Well, this is also a plane electromagnetic wave with amplitude A and angular frequency ω. Ought to be possible just using dimensional analysis to figure out the relationship of A to N and vice versa.

Hmm. Who wanted to know that, anyway? The maniacal Weiss? The judge? (What judge? I seem to remember a judge.) Anyway, it's a rather odd question if the goal is to learn quantum field theory, as Weiss apparently wanted. While it should indeed be possible to answer it using only dimensional analysis, that would sort of miss the interesting aspect: namely, that electromagnetic field strength and photon number are probably noncommuting observables, so an electromagnetic plane wave with a precisely measured amplitude probably doesn't have a precise photon density, and conversely. Maybe Weiss meant the *average* photon density.

Let's see if I can work out the answer using just dimensional analysis, the way Weiss wanted. Then if he kidnaps me again I can tell him about coherent states and all that *if he promises to pay for it*.

This is not gonna be very precise, after being dosed with nitrous and then knocked over the head.

First let's think of light as being classical. Then the energy density is $(\mathbf{E}^2 + \mathbf{B}^2)/2$, modulo a possible factor of 4π that comes from using funny units. In a plane wave of light $\mathbf{E}$ and $\mathbf{B}$ keep wiggling back and forth and turning into each other, but the overall size of either of them is the amplitude A. So the energy density is something like A^2. Maybe there's supposed to be a 1/2 or something somewhere, since neither the $\mathbf{E}$ nor the $\mathbf{B}$ field is equal to its maximum A *all* the time, but let's not worry about that... we'll be glad if we get anything close to the right answer. Okay, so A^2 is the energy density, measured in something like joules per cubic meter.

On the other hand, let's think of light as being made of photons. The energy of a photon is $E = \hbar\omega$ where $\hbar$ is Planck's constant and ω is the angular frequency, not the number of full wiggles per second but that times 2π (which is why $\hbar = h/2\pi$). So the energy density should be $\hbar\omega d$, where d is the average number of photons per cubic meter. So we should have

$$A^2 = \hbar\omega d$$

Does that make sense? The lower the frequency, the more photons we need to get a particular energy, so that part makes sense. It's sort of odd how the number of photons is proportional to the *square* of the amplitude; you mighta thunk it would just be proportional.

Hmm, this is sort of like a physics qualifying exam... which is one of the reasons I didn't go into physics... all those questions like "Say you drop a pion from a height of 2 meters in an external magnetic field of 40 Gauss pointing along the y axis. How high will the pion bounce and what is its charge?" So full of weird kinds of intuitive reasoning, so much harder than proving theorems.

Anyway, let's see, for some reason Weiss wanted to know the density N of photons per square meter per second, instead of per cubic meter. Since photons move at the speed of light, I guess "per second" is the same as "c times per meter" here, so $N = cd$, so

$$A^2 = \hbar\omega N/c.$$

Hmm. So what might be fun is to remember how coherent states work and see why the state that looks like a plane wave of amplitude A has average photon density proportional to $\sqrt{A}$, if that's actually true.

6 John Baez: A Six-Step Program

I suggested:

> Let's work out the QFT description of an electromagnetic plane wave, and see how many photons it has in it (on average).

and Michael Weiss wrote:

> I thought we did the "how many photons on average" part already: A^2/ω per unit volume.

We figured that out by a combination of black magic and seat-of-the-pants intuition. Namely, we took the classical formula for the energy density of the electromagnetic field, and the quantum formula for the energy of a photon of a given frequency, combined them, and used that to guess the photon density of a plane wave. This is known as "semiclassical" (i.e., half-assed) reasoning. It'll probably give the right answer here, but it's not as satisfying as doing it all properly using quantum field theory.

So let's work it out using quantum field theory! It's always crucial, when doing an involved calculation, to know the answer ahead of time. Now that we know the answer to "what's the photon density of the quantum-field-theoretic description of an electromagnetic plane wave", we are in a good position to derive the answer in a more careful way.

This should be fun, because it's *really not so obvious* to me why the answer is what it is. In redoing the computation carefully, we can try to make it obvious why the answer is A^2/ω (if it really is.)

> For example, when I start picturing an electromagnetic plane wave, I say, OK, that's $Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. And I picture stacked planes, and I remember that this is really 4d spacetime and $(\mathbf{k},\omega)$ are really the coordinates of a 1-form. Hmm, what about A? Well, there's a vector field on each of the stacked planes. Gee, I should really somehow drag in the 2-forminess of $(\mathbf{E},\mathbf{B})$ and that fact that it's the differential of the vector potential, and the $U(1)$ gauge and all that, but I can't picture any of that stuff yet. Also there's something wrong with saying that $Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ is a vector. I think I complexified somewhere without realizing it.

Let's see, I think you are trying to do too many things at once here. You are trying to understand classical electromagnetism while simultaneously trying to understand coherent states. I suggest the following game plan:

1. understand coherent states for a harmonic oscillator with one degree of freedom.
2. understand coherent states for a harmonic oscillator with n degrees of freedom.
3. understand coherent states for a free massless scalar field in one dimensional space.
4. understand coherent states for a free massless scalar field in 3 dimensional space.
5. undestand how the equation of a free massless scalar field is related to that of a free massless spin-1 field, i.e. electromagnetism.
6. understand coherent states for the electromagnetic field.

This may seem to multiply our difficulties by six, but I really think it will make the problem much easier. Break it down into bite-sized pieces! Steps 1–4 are a nice gentle ramp, and then mixing in the answer to 5 it should be a snap to get 6.

7 John Baez: A Gaussian Bump

Let me sketch step 1 here and see if I can get Michael to do the actual work.

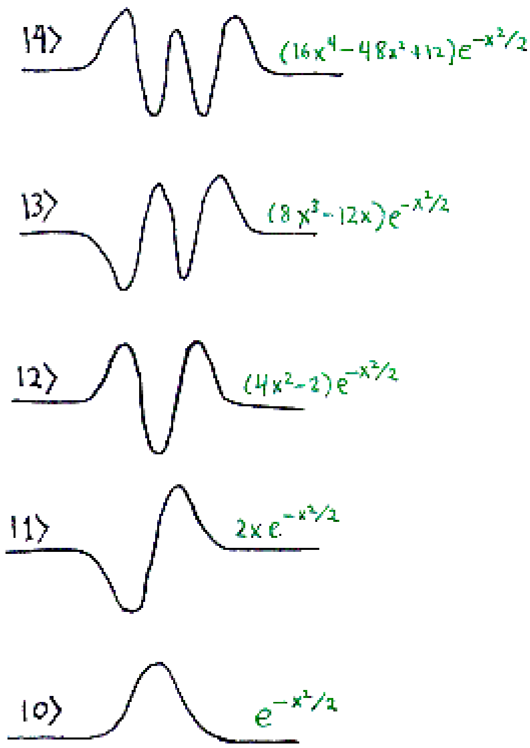
We can think of states of the harmonic oscillator as wavefunctions, complex functions on the line, but they have a basis given by eigenstates of the harmonic oscillator Hamiltonian H. We call these states $|n\rangle$ where $n = 0, 1, 2, 3$, etc., and we have

$$H|n\rangle = (n + 1/2)|n\rangle$$

As wavefunctions, $|0\rangle$ is a Gaussian bump centered at the origin. This is the "ground state" of the harmonic oscillator, the state with least energy. The state $|n\rangle$ is the same Gaussian bump times a polynomial of degree n, giving a function whose graph crosses the x axis n times.

We can think of $|n\rangle$ as a state with n "quanta" in it. Quanta of what? Quanta of energy! This is a little weird, but it'll come in handy to think of this way later. By the time we get to step 6, these "quanta" will be honest-to-goodness photons.

So it's nice to have operators that create and destroy quanta. We'll use the usual annihilation operator a and creation operator a^*, given by



Eigenstates of the harmonic oscillator

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

and

$$a^*|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

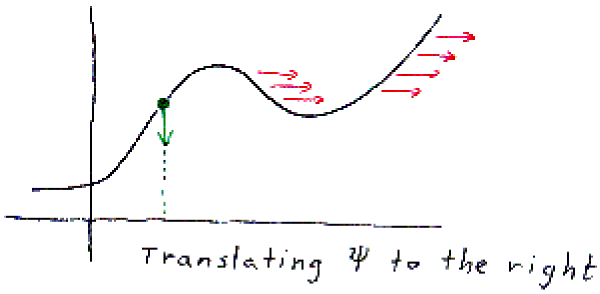
One can relate these guys to the momentum and position operators p and q, which act on wavefunctions as follows:

$$p = -i\frac{d}{dx}$$
$$q = x$$

In the latter equation I really mean "q is multiplication by the function x"; these equations make sense if you apply both sides to some wavefunction.

So maybe Michael can remember or figure out the formulas relating the p's and q's to the a's and a^*'s.

Once we have those, there's something fun we can do.



To translate a wavefunction ψ to the right by some amount c, all we need to do is apply the operator

$$e^{-ipc}$$

to it. The reason is that

$$\begin{aligned}\left(\frac{d}{dc}e^{-ipc}\psi\right)(x) &= (-ipe^{-ipc}\psi)(x) \\ &= -\left(\frac{d}{dx}e^{-ipc}\psi\right)(x)\end{aligned}$$

so the rate at which $e^{-ipc}\psi$ changes as we change c is really just minus the derivative of that function... meaning that it's getting translated over to the right. (We'll give some more detail for this step at the end of this section.) Folks say that the momentum operator p is the "generator of translations".

So we can get a wavefunction that's a Gaussian bump centered at the point $x = c$ by taking our ground state $|0\rangle$ and translating it, getting:

$$e^{-ipc}|0\rangle$$

This is called a "coherent state". In some sense it's the best quantum approximation to a classical state of the harmonic oscillator where the momentum is zero and the position is c. (We can make this more precise later if desired.)

If we express p in terms of a and a^*, and write

$$e^{-ipc} = 1 - ipc + \frac{(ipc)^2}{2!} + \dots$$

we can expand our coherent state in terms of the eigenstates $|n\rangle$. What does it look like?

If we figure this out, we can see what is the expected number of "quanta" in the coherent state. And this will eventually let us figure out the expected number of photons in a coherent state of the electromagnetic field: for example, a state which is the best quantum approximation to a plane wave solution of the classical Maxwell equations. It looks like there should should be about c^2 "quanta" in the coherent state $e^{-ipc}|0\rangle$. This should shed some more light (pardon the pun) on why our previous computations gave a photon density proportional to the amplitude squared and thus the energy density.

The thing to understand is why, even when we have a whole bunch of photons presumably in phase and adding up to a monochromatic beam of light, the amplitude is only proportional to the square root of the photon number. You could easily imagine that a bunch of photons completely randomly out of phase would give an average amplitude proportional to the square root of the photon number, just as $|\text{heads} - \text{tails}|$ grows on average like the number of coins tossed (for a fair coin).

A few more details:

Suppose we have a wavefunction ψ. What is $e^{-icp}\psi$? The answer is: it's just ψ translated c units to the right.

Why? If we take ψ and translate it c units to the right we get

$$\psi(x - c)$$

so we need to show that

$$(e^{-icp}\psi)(x) = \psi(x - c).$$

To show this, first note that it's obviously true when $c = 1$. Then take the derivative of both sides as a function of c and note that they are equal. That does the job.

We are assuming that if two differentiable functions are equal somewhere and their derivatives agree everywhere, then they can't "start being different", so they must be equal everywhere.

Or if that sounds too vague:

Technically, we are just using the fundamental theorem of calculus. Say we have two differentiable functions $f(s)$ and $g(s)$. Then

$$f(x) = f(0) + \int_0^x f'(s)ds$$

It follows from this that if $f(0) = g(0)$ and $f'(s) = g'(s)$ for all s, then $f(x) = g(x)$ for all x.

8 Michael Weiss: Hmm, homework on the weekend!

Hmm, homework on the weekend! *(grumble, grumble)* Oh, well, beats cleaning out the attic ("Whaddya mean we have wasps!? Sure they're not photons?")

John Baez assigns:

> So maybe Michael can remember or figure out the formulas relating the p's and q's to the a's and a^*'s.

Hey, I know that! Basic QM. . . just toss off the answer, $q + ip$, $q - ip$,. . . No, wait, what does it say here: "Show all work!!"

All righty, then. Actually this is kind of fun. I still remember how slick and utterly unintuitive this seemed when I first encountered it in Dirac's book.

Moreover, this slick calculation evokes even earlier memories, of two jewels of high-school algebra: $(a + b)(a - b) = a^2 - b^2$, and $(a + bi)(a - bi) = a^2 + b^2$. If these have lost their luster over the years, try them out on your favorite bright

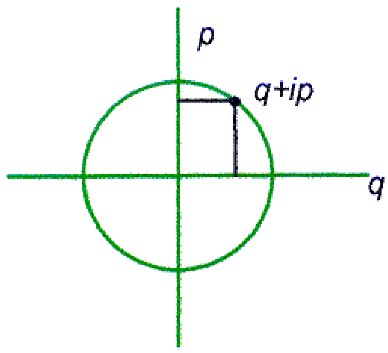
8-year old. I think of $(a + b)(a - b)$ as the primordial instance of that staple of mathematical physics: cancellation of terms, leaving a simple final result.

Well, if having the cross-term cancel is cool, having them *not* cancel turns out to be even cooler!

Dirac does offer one slender reed of motivation, a single sentence: "The above equations are all as in the classical theory." The ultimate incarnation of Bohr's correspondence principle. In fact, Olivier Darrigol wrote a book about that a few years back:

> *From C-Numbers to Q-Numbers: The Classical Analogy in the History of Quantum Theory*. California Studies in the History of Science, University of California Press, December 1992.

OK, so let's start with a classical dot racing around in a circle. Projecting onto the x-axis gives simple harmonic motion. We'll use q for the x-coordinate and p for the y-coordinate, following Hamiltonian tradition. (Why p's and q's? Because Hamilton hung out in Irish pubs, minding his pints and quarts?)



Now as complex number, $q + ip$ is that racing dot. Everything flows from $q + ip$. Let me be ruthless with factors of m and the spring constant and the like— set them all to 1. The total energy is $H = (q^2 + p^2)/2$, or $\frac{1}{2}(q + ip)(q - ip)$. As a function of time, $q + ip = e^{-it}$ (for clockwise motion). The velocity vector is $-i(q + ip)$.

OK, so let's do the same thing in Quantum Land. Now q and p no longer commute; in fact, $qp - pq = i$. So we set $Z = q + ip$ and try to compute H, which was $ZZ^*/2$ last time. But this time the cross-terms don't cancel, and we get:

$$\begin{aligned} ZZ^*/2 &= (q^2 + p^2 + 1)/2 &= H + 1/2 \\ Z^*Z/2 &= (q^2 + p^2 - 1)/2 &= H - 1/2 \end{aligned}$$

I guess I won't go through the whole development. To get the annihilation operator a and the creation operator a^*, we just need to absorb that pesky factor of 2 into a and a^*: $a = Z/\sqrt{2}$, $a^* = Z^*/\sqrt{2}$. So we get:

$$\begin{array}{cc} aa^* = H + 1/2 & [a, a^*] = aa^* - a^*a = 1 \\ a^*a = H - 1/2 & [q, p] = qp - pq = i \\ a = (q + ip)/\sqrt{2} & p = (a - a^*)/i\sqrt{2} \\ a^* = (q - ip)/\sqrt{2} & q = (a + a^*)/\sqrt{2} \end{array}$$

I think that will do it for HW, part 1.

Hmmm. John Baez promised that the plain old harmonic oscillator would have something to do with **E** and **B**. Hey wait, the energy for the harmonic oscillator is $\frac{p^2+q^2}{2}$; for the electromagnetic field, $\frac{\mathbf{E}^2+\mathbf{B}^2}{2}$. Coincidence? You be the judge.

OK, part 2 of the homework. First John Baez defines a coherent state to be $e^{-ipc}|0\rangle$, then asks:

> If we express p in terms of a and a^*, and write
>
> $$e^{-ipc} = 1 - ipc + \frac{(ipc)^2}{2!} + \ldots$$
>
> we can expand our coherent state in terms of the eigenstates $|n\rangle$. What does it look like?

Since $p = (a - a^*)/i\sqrt{2}$, $-ipc = -c(a - a^*)/\sqrt{2}$. Let's write D for $c/\sqrt{2}$, so $-ipc = -D(a - a^*)$. So our coherent state is:

$$|0\rangle - \frac{D(a - a^*)}{1!}|0\rangle + \frac{D^2(a - a^*)^2}{2!}|0\rangle - \frac{D^3(a - a^*)^3}{3!}|0\rangle + \ldots$$

The problem is to figure out the coefficient of $|n\rangle$, i.e., express the coherent state in the form:

$$C_0|0\rangle + C_1|1\rangle + C_2|2\rangle + \ldots$$

[At this point, the poor student went off on a long detour. Inspired by John Baez's remarks about coin-tossing, he rambled through the theory of random walks— at random! Alas, this lead nowhere, but (by a process to tortuous to summarize) it did inspire his next step. —*ed.*]

So I guessed that

$$C_n = \frac{D}{\sqrt{n}} C_{n-1}$$

so of course

$$C_n = \frac{D^n}{\sqrt{n!}} C_0$$

Hmmm. Let's use this guess for the C_n, and apply a to $\sum_n C_n |n\rangle$; to make things a little simpler, let's divide through by C_0 first. We get:

$$\begin{aligned} a\left(|0\rangle + \frac{D}{1}|1\rangle + \frac{D^2}{\sqrt{2!}}|2\rangle + \dots\right) &= \\ 0 + \frac{D}{1}|0\rangle + \frac{D^2}{\sqrt{2!}}\sqrt{2}|1\rangle + \dots &= \\ D\left(|0\rangle + \frac{D}{1}|1\rangle + \dots\right) \end{aligned}$$

I.e., the coherent state would be an eigenstate of the annihilation operator a!

So I tried to prove *this* with combinatorial tricks, but no luck. Then I had the idea of going back to using q and p, instead of a and a^*. With hindsight, the suggestion to rewrite $1 - ipc + \frac{(ipc)^2}{2!} - \dots$ in terms of a and a^* looks like a false lead. Say it ain't so, John!

So here's how my solution goes: [slicker solutions will be found later in these notes. —*ed.*]

First off, we need a generalization of $[q, p] = i$. Here it is:

$$[q, p^n] = qp^n - p^n q = nip^{n-1}$$

or

$$qp^n = p^n q + nip^{n-1}$$

which allows us to move q's past p's.

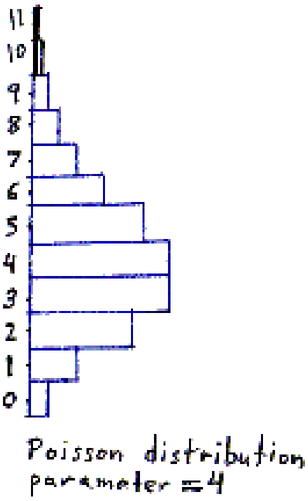
Next we apply this rule to qe^{-ipc}:

$$\begin{aligned} q\left(1 - icp + \frac{(ic)^2}{2!}p^2 - \frac{(ic)^3}{3!}p^3 + \dots\right) &= \left(1 - icp + \frac{(ic)^2}{2!}p^2 - \frac{(ic)^3}{3!}p^3 + \dots\right)q \\ &+ \left(0 - ic1i + \frac{(ic)^2}{2!}2ip - \frac{(ic)^3}{3!}3ip^2 + \dots\right) \end{aligned}$$

or $qe^{-ipc} = e^{-ipc}q + ce^{-ipc}$. So:

$$
\begin{aligned}
(q + ip)e^{-ipc}|0\rangle &= e^{-ipc}(q + ip)|0\rangle + ce^{-ipc}|0\rangle \\
&= ce^{-ipc}|0\rangle
\end{aligned}
$$

since $q + ip$ does annihilate $|0\rangle$. So $e^{-ipc}|0\rangle$ is an eigenstate of a, with eigenvalue $c/\sqrt{2}$ $(= D)$.



And now for the payoff!

If the coherent state is proportional to:

$$
|0\rangle + \frac{D}{\sqrt{1!}}|1\rangle + \frac{D^2}{\sqrt{2!}}|2\rangle + \ldots
$$

then the probability distribution over the energy eigenstates is:

$$
\begin{aligned}
|0\rangle &: e^{-D^2}1 \\
|1\rangle &: e^{-D^2}\frac{D^2}{1!} \\
|2\rangle &: e^{-D^2}\frac{D^4}{2!} \\
\ldots \\
|n\rangle &: e^{-D^2}\frac{D^{2n}}{n!} \\
\ldots
\end{aligned}
$$

that is, a Poisson distribution with parameter D^2. (The e^{-D^2} is just the normalization factor, to make the probabilities add up to 1.) The expectation value for this Poisson distribution is D^2.

In other words, the average number of photons—err, quanta— in the coherent state is D^2.

Hmm, Baez has already pointed out how the coherent state *looks* like a bump displaced c units from the origin.

Suddenly the "classical dot" racing in a circle has a whole new lease on life. There *was* something pretty unsavory about what Dirac did to it. It reminds me of those "before and after" pictures— are they really the same person? Before: the dot races in a circle of *any radius*. After: nothing races around at all, instead we have a bunch of wavefunctions that just *sit there*. Yeah, I know— only after we've factored out the time-dependent part $e^{-i(n+1/2)t}$. But this kind of motion is in an imaginary direction, and doesn't affect the *wherabouts* of the particle, if you know what I mean.

Could these utterly different kinematics really come from the same cinematographer?

Sure! Just use coherent states. As D increases continuously, the average energy increases continuously, as does the range of motion of the bump.

Hmm. Guess I should check that last assertion.

Hokay, we want the time evolution of the bump. Should be a piece of cake. The time evolution of $|n\rangle$ is just $e^{-i(n+1/2)t}|n\rangle$, and we've expressed the coherent state as a sum of energy eigenstates.

Let's redefine the zero-point of energy to get rid of that annoying $\frac{1}{2}$. So $e^{-iHt}e^{-ipc}|0\rangle$ is equal to C_0 times:

$$|0\rangle + e^{-it}\frac{D}{\sqrt{1!}}|1\rangle + e^{-2it}\frac{D^2}{\sqrt{2!}}|2\rangle + \dots$$

that is, $\exp(-ipe^{-it}c)|0\rangle$, since basically we've just replaced D with $e^{-it}D$, and c with $e^{-it}c$. (The argument above didn't actually *use* the assumption that c was real anywhere.)

Hmmm. A bump that swang back and forth harmonically would look like this: $\exp(-ip(\cos t)c)|0\rangle$. How to relate this to $\exp(-ipe^{-it}c)|0\rangle$?

Aha! Euler to the rescue: $e^{-it} = \cos t - i \sin t$. Uh-oh, now we've got $e^{-ip(\cos t)c}e^{-p(\sin t)c}|0\rangle$.

What does $e^{-p(\sin t)c}|0\rangle$ look like?

Help!

9 John Baez: Ever wonder why they call it "phase space"?

> OK, so let's start with a classical dot racing around in a circle. Projecting onto the x-axis gives simple harmonic motion. We'll use q for the x-coordinate and p for the y-coordinate, following Hamiltonian tradition.

Of course, by "x-coordinate" you mean "position" and by "y-coordinate" you mean "momentum". We've got here a point in phase space, oscillating harmonically. And as you note, one key to understanding quantum mechanics is to see this phase space as the complex plane!

> Now as complex number, $q + ip$ is that racing dot. Everything flows from $q + ip$.

Indeed. Ever wonder why they call it "phase space"? I don't know the history, but here we see a damn good reason: as our point $Z = q + ip$ circles the origin, nothing changes but its *phase*.

The whole point of coherent states is to see very clearly what happens to this picture when we go into quantum mechanics. Of course, everybody knows that when p and q become operators, we can make Z and its complex conjugate into operators too, which are basically just the creation and annihilation operators:

$$a = \frac{Z}{\sqrt{2}}, \qquad a^* = \frac{Z^*}{\sqrt{2}}$$

But the cool part is that there are also *states* that are quantum analogues of points circling the origin: the coherent states. As you note, these are just the eigenstates of the annihilation operator. But I prefer to visualize them as Gaussian wavefunctions: a kind of blurred-out version of a state in which a particle has a definite position and momentum. If you start with such a state, and evolve it in time, its (expectation value of) position and momentum oscillate just like that of a classical particle, and if I remember correctly, they maintain a basically Gaussian shape, though probably with some funny complex phase factor stuff thrown in…

> …nothing races around at all, instead we have a bunch of wavefunctions that just *sit there*…

Yeah, sure, the *eigenstates* of the Hamiltonian just sit there, after we ignore the time-dependent phase. But these states are very unlike the classical states we know and love. I like to pose the following puzzle to kids just learning quantum mechanics:

"Take this eraser. [I brandish my eraser threateningly as I stand before the blackboard.] Put it into an eigenstate of the Hamiltonian. Now it's in a stationary state! It doesn't *do anything* as time passes. It just sits there, except for a time-dependent phase! [I demonstrate an eraser nonchalantly hovering in midair, only its phase wiggling slightly.] So what does this mean, that you can levitate an eraser just by putting it into an eigenstate?"

And of course they eventually get the point: it's not so easy in practice to put anything big into an eigenstate of the Hamiltonian. It's the *coherent states* that are close to the classical physics we know and love.

> Could these utterly different kinematics really come from the same cinematographer?
>
> Sure! Just use coherent states. As D increases continuously, the average energy increases continuously, as does the range of motion of the bump.
>
> Hmm. Guess I should check that last assertion.

That should be true. Remember, the expected value of the Hamiltonian is

$$\langle H \rangle = \frac{\langle p^2 \rangle + \langle q^2 \rangle}{2}$$

As we take our basic Gaussian bump (the ground state) and translate it, $\langle p^2 \rangle$ stays the same, since its shape stays the same. $\langle q^2 \rangle$, on the other hand, gets bigger, basically because the average value of q gets bigger (though I'm being a bit rough here). So we can tune $\langle H \rangle$ to whatever value we want. . . at least for values bigger than the "zero-point energy", which is $\frac{1}{2}$.

10 John Baez: Perhaps the nicest Gaussian of all

Let's see. In my first reply to Michael Weiss' post I said a few words about coherent states. Let me say a few more general remarks about coherent states. Eventually in some later post I will turn towards the practical business at hand: calculating what coherent states look like in the basis given by eigenstates of the harmonic oscillator Hamiltonian. (Michael already did this, so I'll just say a bit about what he did.)

A coherent state is supposed to be roughly a "best quantum approximation to a classical state". There is a big theory of coherent states which makes this a lot more precise, but let's not get into that. Instead, let's just ponder what this might mean.

Classically, a particle on the line has a definite position q and momentum p, so it is described by a point in phase space, (q, p). Quantum mechanically, the more we know about position, the less we know about momentum, and vice versa. Our ability to know both at once is limited by Heisenberg's uncertainty principle:

$$\Delta p \, \Delta q \geq \frac{1}{2}$$

(Here, as always in this thread, we are working in units where $\hbar$ is one, as well as the mass of our particle and the spring constant of our harmonic oscillator.)

Which states do the best job of simultaneously minimizing the uncertainty in position and momentum? Which states make $\Delta p \, \Delta q$ *equal* to $\frac{1}{2}$? It turns out that these states are precisely the Gaussians, possibly translated, and possibly multiplied by a complex exponential.

Perhaps the nicest Gaussian of all is

$$e^{-x^2/2}$$

since this is the ground state of the harmonic oscillator Hamiltonian, at least after we normalize it.

This function is its own Fourier transform (if we define our Fourier transform right). Since you can compute the uncertainty in momentum by taking the Fourier transform of your wavefunction and then computing the uncertainty in position of *that*, this Gaussian must have the same uncertainty in position as it does in momentum. If everything I've said so far is true, we must therefore have

$$\Delta p = \Delta q = \frac{1}{\sqrt{2}}$$

for this Gaussian.

Of course, there are lots of other Gaussians centered at the origin with $\Delta p \, \Delta q = \frac{1}{2}$. We can squish our Gaussian or stretch it out like this:

$$e^{-x^2/2\sigma}$$

The Fourier transform of a squished-in skinny Gaussian is a stretched-out squat Gaussian and vice versa. So all these Gaussians have

$$\Delta p\,\Delta q = \frac{1}{2}$$

but the one we chose is the only one where

$$\Delta p = \Delta q$$

That can be our definition of a "coherent state": a state that simultaneously minimizes the uncertainty in position and momentum, and makes these uncertainties *equal* each other. Later we can think more about "squeezed states" where the uncertainties are not equal. There was recently a big fad where everyone was making squeezed states of light in the lab. But for now, we will not squeeze our states.

So: the Gaussian

$$e^{-x^2/2}$$

is the primordial "coherent state". In this state, the expectation value of position is obviously zero, since the bump is symmetrically centered at the origin. The expectation value of momentum is also obviously zero, since:

1) the Fourier transform of this function is itself, so whatever applies to position applies to momentum as well,

or if you prefer,

2) the expectation value of momentum is zero for any *real-valued* wavefunction. (Hint: to see this, just use integration by parts.)

So $e^{-x^2/2}$ is a coherent state with expectation values

$$\langle p\rangle = \langle q\rangle = 0.$$

We can get lots more coherent states by taking this Gaussian and translating it in position space and/or momentum space. Translating in position space by c, we get a Gaussian

$$e^{-(x-c)^2/2}$$

This is the coherent state I wanted Michael to express in terms of eigenstates of the harmonic oscillator Hamiltonian. This obviously has

$$\langle q\rangle = c$$

and since it's real-valued it still has

$$\langle p\rangle = 0.$$

Translating in momentum space by some amount b is the same as multiplying by a complex exponential e^{ibx}. Or, if you prefer, just take a Fourier transform, translate by b, and take an inverse Fourier transform. Same thing. If we do this to our primordial Gaussian bump, we get

$$e^{ibx-x^2/2}$$

which is our coherent state with

$$\langle q\rangle = 0$$

and

$$\langle p\rangle = b.$$

Why does it still have $\langle q\rangle = 0$? Well, we are just multiplying our Gaussian bump by a unit complex number or "phase" at each point, and this doesn't affect the expectation value of position.

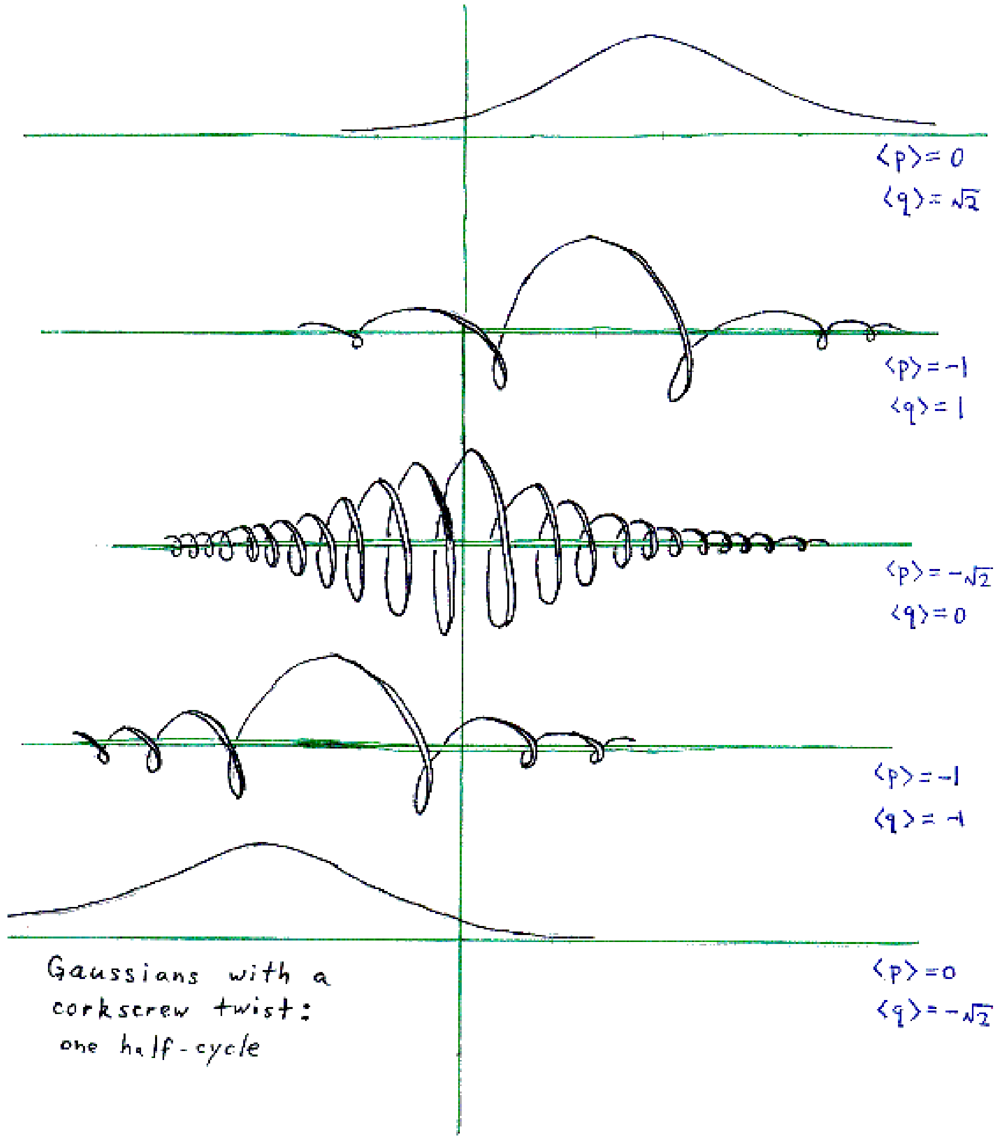
Finally, we can translate in *both* position *and* momentum space directions. These two operations don't commute, of course, since the position and momentum operators don't commute, and momentum is the generator of translations in position space, while position is the generator of translations in momentum space (possibly up to an annoying minus sign).

What do we get? Well, take our bump and first translate it in position space by c:

$$e^{-(x-c)^2/2}$$

and then in momentum space by b:

$$e^{ibx-(x-c^2/2)}$$


$\langle p \rangle = 0$
$\langle q \rangle = \sqrt{2}$
$\langle p \rangle = -1$
$\langle q \rangle = 1$
$\langle p \rangle = -\sqrt{2}$
$\langle q \rangle = 0$
$\langle p \rangle = -1$
$\langle q \rangle = -1$
$\langle p \rangle = 0$
$\langle q \rangle = -\sqrt{2}$
Gaussians with a
corkscrew twist:
one half-cycle

Or, alternatively, first translate it in momentum space by b:

$$e^{ibx-x^2/2}$$

and then in position space by c:

$$e^{ib(x-c)-(x-c)^2/2}$$

I claim that these are equal *up to a phase*...one of them is e^{ibc} times the other. This is always how translations in position space and momentum space fail to commute.

While terribly important in some ways, the phase is not such a big deal in other ways. (That's the weird thing about quantum mechanics when you are first learning it: sometimes a phase is very important, while other times it doesn't matter at all. Of course, it just depends what you're doing.) A phase times a coherent state is still a coherent state in my book. So we have gotten our hands on a coherent state with

$$\langle p\rangle = b$$

and

$$\langle q\rangle = c$$

A translated Gaussian bump, with a corkscrew twist thrown in! I hope you *visualize* this thing for various values of b and c; it's a very pretty thing, and it will serve as our quantum "best approximation" to a particle with momentum b and position c.

Then, later, we will use a souped-up version of this as the quantum- field-theoretic "best approximation" to a particular state of a classical field theory, like electromagnetism.

11 John Baez: The Most Enlightening Set of Names

Before I dig into the business of working out what coherent states look like in the basis of eigenstates of the harmonic oscillator Hamiltonian, let me comment on one thing:

> Hmmm. John Baez promised that the plain old harmonic oscillator would have something to do with **E** and **B**. Hey wait, the energy for

the harmonic for the harmonic oscillator is $\frac{p^2+q^2}{2}$; for the electromagnetic field, $\frac{\mathbf{E}^2+\mathbf{B}^2}{2}$. Coincidence? You be the judge.

Of course it's no coincidence; the electromagnetic field is just a big bunch of harmonic oscillators, one for each "mode", and the formula for the energy is just like that for the harmonic oscillator. **B** is sort of like the "position" and **E** is sort of like the "momentum". But let's leave the details for later on.

What we're doing now is like finishing school. Everyone bumps into the harmonic oscillator in quantum mechanics, but rather few get to see its full beauty. There are three main representations to learn:

1. the Schrödinger representation
2. the Heisenberg (or Fock) representation
3. the Bargmann-Segal representation

The first should really be called the "wave" representation. In this representation we diagonalize the position operator, thinking of the state as a function on position space.

Similarly, the second should be called the "particle" representation. In this we diagonalize the energy, thinking of the state as a linear combination of states $|n\rangle$ having n "quanta" of energy in them.

The equivalence of these first two representations is the basis of "wave-particle duality" in quantum field theory.

The third representation could be called the "complex wave" representation. At least that's what it's called in the book *Introduction to Algebraic and Constructive Quantum Field Theory*, where the first representation is called the "real wave representation". In some rough sense, this representation diagonalizes the creation operators. Of course, not being self-adjoint, the creation operators can't be diagonalized in the usual sense. There is a nice substitute, however. In the complex wave representation, we think of phase space as a complex vector space using the trick

$$Z = q + ip$$

and then think of states as *analytic* functions on phase space. Then the creation operator becomes multiplication by Z.

Perhaps this would be the most enlightening set of names for these three representations:

1. the configuration space representation
2. the particle representation
3. the phase space representation

Everyone who studies quantum mechanics learns about the first two representations. The third, while in many ways the most beautiful, is somewhat less widely known. We won't get into it much here. The only reason I mention it is that it's lurking in the background whenever you relate quantum mechanics to the classical phase space and drag in the $Z = q + ip$ trick.

Anyway, if you master these three basic viewpoints on the harmonic oscillator, it's a snap to generalize to quantum field theory, at least for free quantum fields, which are just big bunches of harmonic oscillators.

12 Michael Weiss: Roll over Beethoven

John Baez writes:

> Before I dig into the business of working out what coherent states look like in the basis of eigenstates of the harmonic oscillator Hamiltonian, let me comment on one thing:

Uh-oh, better get the rest of my homework in before the professor goes over the assignment in class.

OK, last time I figured out that the coherent state you get by sliding the Gaussian bump c units to the right:

$$e^{-icp}|0\rangle$$

is proportional to this, in the basis of eigenstates of the Hamiltonian, aka "particle representation":

$$e^{-D^2/2} \sum_n \frac{D^n}{\sqrt{n!}} |n\rangle$$

where $D = c/\sqrt{2}$. This time I've included the factor $e^{-D^2/2}$, so as to get a normalized state-vector. Since e^{-icp} is a unitary operator, by some theorem or other, $e^{-icp}|0\rangle$ is also normalized. So we've expressed the coherent state in the particle representation, up to a phase.

The probability distribution of this state-vector is a Poisson distribution with mean value D^2:

$$\text{Prob(we're in state } |n\rangle) = e^{-D^2}\frac{(D^2)^n}{n!}$$

$$\text{Mean value} = \sum_n n\,\text{Prob(we're in state } |n\rangle)$$

(To do the mean-value sum, notice that the $n = 0$ term politely disappears, and if you pull out a factor of D^2, what's left is just the sum of all the probabilities—which had better add up to 1.)

Now this is nice. The original question was:

> "If an electromagnetic wave has amplitude A and angular frequency ω, how many photons does it contain per unit volume, on the average?"

and dimensional analysis said: A^2/ω. Well, presumably c and D will end up being proportional to A, once we actually start talking about electromagnetic waves, and not this mickey-mouse-dot racing around a circle! And the mean value should be proportional to the photon density. So we have our explanation for the A^2 factor.

But to make this convincing, we have to know that D really *does* correspond to the radius of the dot's racetrack. So we need the time-evolution of the coherent state.

Now here I got stuck for a while, for I ignored the old adage: "Never compute anything in physics unless you already know the answer!" (Who said that, Wheeler?) I was trying to show that $e^{-icp}|0\rangle$ would evolve to $e^{-icp\cos t}|0\rangle$, since that's the formula for simple harmonic motion.

No good! The coherent state wavefunction has both position *and* momentum encoded in it. You can't expect the position to change without the momentum *also* changing.

OK, let's start over. First, we have to consider a more general coherent state, say one with momentum b and position c, thus:

$$e^{ibq}e^{-icp}|0\rangle$$

which I'll call Coh1($c + ib$). [Coh1 because it's "coherent state, take 1": John Baez will shortly introduce a better choice. —*ed.*]

Now we want to express this in the particle representation, because we know how the states $|n\rangle$ evolve: $|n\rangle$ evolves to $e^{-int}|n\rangle$ (if we factor out the common $e^{-it/2}$, i.e., redefine the energy zero-point).

I did this for the special coherent state Coh1(c) a while back, but I was working entirely too hard. I used the fact:

$$[q, p^n] = inp^{n-1}$$

Now this looks a lot like a derivative formula: $[q, p^n] = i\frac{dp^n}{dp}$. (Ignore the fact that I haven't defined d/dp.) Actually, this shouldn't be surprising: we know that

$$[p, A] = -i\frac{dA}{dx}$$

i.e., bracketing with p is just about the same as taking derivatives with respect to x. Now q and x are pretty closely related, and what holds for q ought to hold for p, using Fourier transforms and all.

So we should have:

$$\begin{aligned} [p, f(q)] &= -idf/dq \\ [q, g(p)] &= idg/dp \end{aligned}$$

at least if $f(q)$ is a power series in q, and $g(p)$ is a power series in p.

As a check, let's verify the product rule. If that works, then we should have our result for all powers of p and q just by induction, and then for all power series by continuity arguments. The continuity arguments might take up a chapter or two in a functional analysis textbook, but hey, I'm sure the kindly moderator will cut us some slack. *(Pause for thunderbolts to dissipate.)*

$$[s, AB] = sAB - ABs$$

$$\begin{aligned} [s, A]B + A[s, B] &= (sA - As)B + A(sB - Bs) \\ &= sAB - AsB + AsB - ABs \end{aligned}$$

It works!

So:

$$[q, e^{-icp}] = i(-ic)e^{-icp} = ce^{-icp}$$

$$[p, e^{ibq}] = -i(ib)e^{ibq} = be^{ibq}$$

$$[(q + ip), e^{ibq}e^{-icp}] = (c + ib)e^{ibq}e^{-ipc}$$

You have to be a little bit careful here. $e^A e^B$ is *not* generally equal to e^{A+B} if A and B don't commute.

If that last equation doesn't leap out at you from the previous two, well it's just a bit of straightforward grinding.

And so:

$$\begin{aligned}(q+ip)\mathrm{Coh1}(c+ib) &= (q+ip)e^{ibq}e^{-icp}|0\rangle \\ &= e^{ibq}e^{-icp}(q+ip)|0\rangle + (c+ib)e^{ibq}e^{-icp}|0\rangle \\ &= (c+ib)\mathrm{Coh1}(c+ib)\end{aligned}$$

So $\mathrm{Coh1}(c+ib)$ is an eigenvector of $q+ip$ with eigenvalue $c+ib$. It then follows that, up to a phase (call it ι):

$$\mathrm{Coh1}(c+ib) = \iota e^{-(c^2+b^2)/2} \sum_n \frac{(c+ib)^n}{\sqrt{n!}}|n\rangle$$

Remember now that we want to time-evolve this. (So much physics, so little $e^{-iHt}\ldots$) As I said earlier, $|n\rangle$ evolves to $e^{-int}|n\rangle$ (if choose our energy zero-point so as to get rid of the vacuum energy).

So the n-th term of our formula for $\mathrm{Coh1}(c+ib)$ will acquire the factor e^{-int} in t-seconds. But we can absorb this into the factor $(c+ib)^n$, just by replacing $c+ib$ with $(c+ib)e^{-it}$. So:

$$e^{-iHt}\mathrm{Coh1}(c+ib) = \mathrm{Coh1}(e^{-it}(c+ib))$$

Roll over Beethoven, how classical can you get! If I told you that $\mathrm{Coh1}(c+ib)$ was my symbol for a dot in the complex plane at position $c+ib$, you'd say the equation I just wrote is obvious.

[Moderator's note: The aphorism, "Never calculate anything until you know the answer," is indeed due to Wheeler. It appears in Taylor and Wheeler's book *Spacetime Physics* under the name of "Wheeler's First Moral Principle." No other moral principles are mentioned, so maybe it's Wheeler's Only Moral Principle. —Ted Bunn]

13 John Baez: A Gaussian bump with a corkscrew twist!

> OK, last time I figured out that the coherent state you get by sliding the Gaussian bump c units to the right ...

Very nice. I would like to understand this better and think more about the best way of deriving it. The way I suggested to you was stupid and grungy, but you seem to have fought your way through and then discovered some much nicer approaches. I haven't thought about this stuff enough, so I'd like to polish your work to a fine sheen before moving on.

So... let me go back to our general picture of coherent state.

What's the best quantum approximation to a classical particle on the line with specified position and momentum? A Gaussian bump with a corkscrew twist! We will only be interested here in bumps that have equal uncertainty in position and momentum:

$$\Delta p = \Delta q = \frac{1}{2}$$

The simplest case is when

$$\langle p \rangle = \langle q \rangle = 0$$

Then we use the ground state of the harmonic oscillator:

$$|0\rangle = e^{-x^2/2}$$

where I left out the normalization factor to reduce clutter.

To get coherent states with other expectation values of position and momentum, say

$$\langle p \rangle = b, \quad \langle q \rangle = c$$

we can take our ground state, translate it in position space by an amount c, and then translate it in momentum space by an amount b:

$$e^{ibq} e^{-icp} |0\rangle = e^{ibx-(x-c)^2}$$

where I have attempted to make an even number of sign errors. [An allusion to Dirac's comment on the first presentation of the Klein-Nishina formula (by Nishina). See Gamow, *Thirty Years That Shook Physics*. —*ed.*]

But wait! We could also have translated it *first* in momentum space and *then* in position space, getting

$$e^{-icp}e^{ibq}|0\rangle = e^{ib(x-c)-(x-c)^2}$$

How does this answer fit with the other? Well, it differs only by a phase.

"Only a phase"... ah, what an understatement! When physicists and mathematicians mutter darkly about "cocycles", "projective representations", "double covers", "central extensions", and even more intimidating things like "anomalies", "the Virasoro algebra" and "affine Lie algebras", they are secretly complaining about the many subtleties that can caused by a mere phase!

So let us think about this a little bit. The two coherent states above differ by the phase e^{-ibc}. That should be no surprise; the Heisenberg commutations relations

$$pq - qp = -i$$

lead directly — with a dose of mathematical optimism — to the exponentiated version called the "Weyl commutation relations"

$$e^{-icp}e^{ibq} = e^{-ibc}e^{ibq}e^{-icp}$$

which describe how translations in position space and momentum space commute only up to a phase. Actually, mathematical physicists of the rigorous variety prefer to take the Weyl relations as basic and derive the Heisenberg relations as a consequence. But we are being relaxed here so we can think of them as two ways of saying the same thing.

Now, Michael has taken

$$\mathrm{Coh1}(c + ib) = e^{ibq}e^{-icp}|0\rangle$$

as his definition of a coherent state with expected momentum b and expected position c. This is fine... up to a phase... but it's slightly annoying how one needs to "break the symmetry" between momentum and position in this definition. Why not

$$\mathrm{Coh2}(c+ib) = e^{-icp}e^{ibq}|0\rangle \quad ?$$

Or even better, how about some choice that treats position and momentum even-handedly! "Mind your p's and q's!" There's much wisdom in that phrase…

Here's a nice way to mind our p's and q's; we make the following new definition:

$$\mathrm{Coh}(c+ib) = e^{-icp+ibq}|0\rangle$$

Here we "simultaneously translate in position and momentum space" instead of favoring one or the other. This state is not equal to *either* of the two choices listed above, but again it differs only by a phase.

Why?

Well, one can show that

$$\begin{aligned} e^{-icp+ibq} &= e^{-ibc/2}e^{ibq}e^{-icp} \\ &= e^{+ibc/2}e^{-icp}e^{ibq} \end{aligned}$$

at least if I've not made a sign error. So this new definition "steers a middle course" between the other two choices, phase-wise.

(Also, fans of symplectic geometry will appreciate the funny skew- symmetric quality of the expression $-icp+ibq$ in our new definition. But let's not get into that.)

Here's a little assignment for Michael, or any other students willing to pitch in! Remember that c represents *position* and b above represents *momentum*, so (c, b) represents a point in phase space. Also remember that it's good to think of phase space here as the *complex plane*. So let's define

$$z = c + ib.$$

(Don't confuse this lower-case z with the upper-case Z we had before. The upper-case Z was an exhalted operator; the lower-case z is just a lowly complex number.)

Now: take the expression

$$e^{-icp+ibq}$$

and rewrite it in terms of z and its complex conjugate z^*, while simultaneously rewriting p and q in terms of annihilation and creation operators. Remember that

$$\begin{array}{ll} a = (q + ip)/\sqrt{2} & q = (a + a^*)/\sqrt{2} \\ a^* = (q - ip)/\sqrt{2} & p = (a - a^*)/i\sqrt{2} \end{array}$$

Some nice stuff should happen.

If we do this, we will get a cool expression for our coherent state in terms of annihilation and creation operators applied to the vacuum state. This won't immediately solve all our problems, but it should help us understand a lot about how our coherent states of the harmonic oscillator *evolve in time*.

14 Michael Weiss: Now for a bit of straightforward grinding.

OK, so let's set

$$z = c + ib$$

$$\mathrm{Coh}(c + ib) = e^{-icp+ibq}|0\rangle$$

Now for a bit of straightforward grinding. We want to express $-icp + ibq$ in terms of z and a.

$$\begin{aligned} -icp &= -i\frac{z + z^*}{2}\,\frac{a - a^*}{i\sqrt{2}} \\ &= -\frac{1}{2\sqrt{2}}(za + z^*a - za^* - z^*a^*) \\ ibq &= \frac{z - z^*}{2}\,\frac{a + a^*}{\sqrt{2}} \\ &= \frac{1}{2\sqrt{2}}(za - z^*a + za^* - z^*a^*) \\ -icp + ibq &= \frac{1}{\sqrt{2}}(za^* - z^*a) \end{aligned}$$

15 John Baez: My lysdexia has a variety of origins

[You, dear reader, have been shielded by your valiant editor from the sign errors and inconsistent conventions of the original newgroup thread. At least I hope I've caught them all! But just to make your editor's life difficult, John Baez here uses these very errors as a jumping off point for an informative discussion.—*ed.*]

My lysdexia above has a variety of origins. Indeed, this subject is littered with banana peels on which the unwary can slip, so it is probably pedagogically useful to list them.

The main reason for my slip, writing e^{ipc} instead of e^{-ipc}, was the usual symplectic switcheroo: the momentum operator p generates translations in position space, while the position operator q generates translations in momentum space. More precisely, p generates translations to the *right* in position space, while q generates translations to the *left* in momentum space. This is built into the classical Poisson brackets:

$$\{p, q\} = 1, \qquad \{q, p\} = -1$$

and thus in the commutators of the corresponding operators:

$$[p, q] = -i, \qquad [q, p] = i.$$

where an extra factor of $-i$ is traditionally thrown in to further confuse the uninitiated.

Thus in the state $e^{-icp+ibq}|0\rangle$, the expectation value of position is c and the expectation value of momentum is b.

Second, there is a somewhat arbitrary convention about whether we think of a point in phase space with momentum b and position c and as being the point $c+ib$ in the complex plane, or $b+ic$. Time evolution for the harmonic oscillator amounts to having our complex plane rotate as time passes, and if we use $c + ib$ the plane will rotate *clockwise*, while if we use $b+ic$ the plane will rotate *counterclockwise*. The former means that after a time t, the point $c + ib$ will evolve to

$$e^{-it}(c + ib),$$

while the latter means that after a time t, the point $b + ic$ will evolve to

$$e^{it}(b + ic)$$

The latter seems nicer to me, which is another reason for my slip, but we seem to be working with the former convention.

16 John Baez: Michael has roamed ergodically

Okay, after having caught various slips by the absentminded professor, Michael has shown the following:

Let $z = c + ib$ be a point in the phase space of a particle on the line, corresponding to position c and momentum b.

The coherent state with average position c and momentum b is

$$\begin{aligned}\mathrm{Coh}(z) &= e^{-icp+ibq}|0\rangle \\ &= e^{(za^*-z^*a)/\sqrt{2}}|0\rangle\end{aligned}$$

Very good!

Here's some more homework. Actually, looking back over this thread, I see that Michael has roamed ergodically over the space of ways of thinking of this stuff, and has come very close to almost all possible ways, so this homework is not terribly novel.

A) Use the formula

$$\mathrm{Coh}(z) = e^{(za^*-z^*a)/\sqrt{2}}|0\rangle$$

to get a curiously similar formula involving an exponential of only *creation* operators, applied to the vacuum.

The formula is *something* like

$$\mathrm{Coh}(z) = e^{-|z|^2} e^{za^*}|0\rangle$$

but you'll need to stick in a couple of constant factors here and there.

(Actually Michael has already done something like this, starting from a different angle.)

B) Use the commutation relations between H and a^* to work out

$$e^{itH} a^* e^{-itH}$$

and then

$$e^{itH} e^{za^*} e^{-itH}$$

Together with A), use this to work out the time evolution of the coherent state Coh(z).

C) Show that if we evolve a coherent state over one period of our oscillator — i.e., take $t = 2\pi$ — it does *not* return to the same wavefunction, unlike for the classical oscillator.

This corrects a little mistake of Michael's [HA! see below. —*ed.*] where he claimed that

$$e^{-itH}\mathrm{Coh}(z) = \mathrm{Coh}(e^{-it}z).$$

It's not *quite* so simple and nice. Hint: vacuum energy.

17 Michael Weiss: Okay, thanks, Baker, Campbell, and Hausdorff!

Just stopping by for a sec, to drop off some homework. I'll be by again later for a longer chat.

But first—oh professor, I think you took too much off! You say:

> This corrects a little mistake of Michael's where he claimed that
>
> $$e^{-itH}\mathrm{Coh}(z) = \mathrm{Coh}(e^{-it}z).$$
>
> It's not *quite* so simple and nice. Hint: vacuum energy.

But I explicitly said I was redefining the zero-point of energy to get rid of those pesky factors of $e^{-it/2}$. Hmm, maybe you're saying the formulas are trying to tell me something— that I *shouldn't* monkey around with the zero-point? *(Ominous music wells up on the soundtrack).* (Flash to the final scene in a 50s sci-fi movie, as the grey-haired senior scientist portentously intones, "There are aspects of Nature that we change at our peril. Let this be a Lesson To Us All…")

(Actually, the grey-haired senior scientist has already said something about this zero-point stuff: see http://math.ucr.edu/home/baez/harmonic.html)

Okay.

A) Use the formula

$$\mathrm{Coh}(z) = e^{(za^* - z^*a)/\sqrt{2}}|0\rangle$$

to get a curiously similar formula involving an exponential of only *creation* operators, applied to the vacuum.

Here $z = c + ib$.

I did almost this computation once before, but let's do a quick recap.

First: we want to show that $\mathrm{Coh}(z)$ is an eigenvector of the annihilation operator a. For this we need to compute some commutators, and the slick way is to notice that $[a, \cdot]$ acts like a derivative on many operators. At least this works for power series in a and a^*. Say $f(a, a^*)$ is a power series with complex coefficients. Since $[a, a] = 0$ and $[a, a^*] = 1$, we'll get the right result for $[a, f(a, a^*)]$ with this recipe: compute $(d/dx) f(a, x)$ formally, treating a like a constant; then replace x with a^* in the final result.

Using this rule on $e^{za^* - z^* a}$, we get

$$\left[a, e^{(za^*-z^*a)/\sqrt{2}}\right] = \left[a, \frac{za^* - z^* a}{\sqrt{2}}\right] e^{(za^*-z^*a)/\sqrt{2}} = \frac{z}{\sqrt{2}} e^{(za^*-z^*a)/\sqrt{2}}$$

Now put $|0\rangle$ on the right, we get

$$a\,\mathrm{Coh}(z) = a e^{(za^*-z^*a)/\sqrt{2}}|0\rangle = \frac{z}{\sqrt{2}} e^{(za^*-z^*a)/\sqrt{2}}|0\rangle$$

since a annihilates $|0\rangle$. So $a\,\mathrm{Coh}(z) = z/\sqrt{2}\,\mathrm{Coh}(z)$.

Next we expand $\mathrm{Coh}(z)$ in the basis $|0\rangle, |1\rangle \ldots$. From the eigenvalue equation, we get immediately:

$$\mathrm{Coh}(z) = C_0 \left(|0\rangle + \frac{z}{\sqrt{2}}|1\rangle + \ldots + \frac{(z/\sqrt{2})^n}{\sqrt{n!}}|n\rangle + \ldots \right)$$

where C_0 is the coefficient of $|0\rangle$.

We can evaluate $|C_0|$ pretty easily. The norm squared of $\mathrm{Coh}(z)$ is $|C_0|^2 e^{|z|^2/2}$, from the formula we just got. But $\mathrm{Coh}(z)$ has norm 1. How do I know that? Well, $e^{(za^*-z^*a)/\sqrt{2}}$ is unitary. How do I know that? Well, $(za^* - z^* a)/\sqrt{2}$ is i times a self-adjoint operator (just take the adjoint and see what you get), so by some theorem or other its exponential is unitary.

So $|C_0| = e^{-|z|^2/4}$. So we've determined $e^{(za^*-z^*a)/\sqrt{2}}|0\rangle$ up to a phase (let's call the phase ι):

$$\mathrm{Coh}(z) = \iota e^{-|z|^2/4} \sum_n \frac{(z/\sqrt{2})^n}{\sqrt{n!}}|n\rangle$$

Hmmm, now for a new twist. The professor asked for the answer in terms of a^*. Well, $a^{*^n}|0\rangle = \sqrt{n!}\,|n\rangle$, —hey, this works out nicely:

$$e^{(za^*-z^*a)/\sqrt{2}}|0\rangle = \iota e^{-|z|^2/4} \sum_n \frac{(z/\sqrt{2})^n}{n!} a^{*^n}|0\rangle = \iota e^{-|z|^2/4} e^{za^*/\sqrt{2}}|0\rangle$$

What are we going to do about that phase ι?

Hmmm, let's take another approach. If life was *really* simple, we could just say that $e^{za^*-z^*a} = e^{za^*}e^{-z^*a}$ (*it isn't*), and since a annihilates $|0\rangle$, $e^{-z^*a}|0\rangle = |0\rangle$ (just expand out e^{-z^*a} in a Taylor series). So we'd have:

$$e^{za^*-z^*a}|0\rangle = e^{za^*}|0\rangle \qquad \text{(NOT!!)}$$

But skimming back over the thread, we get strong hints that

$$e^{za^*-z^*a} = e^{\text{number}} e^{za^*} e^{-z^*a}$$

Let's ask. Oh, professor!

> The Baker-Campbell-Hausdorff formula says that when $[A, B]$ commutes with everything
>
> $$e^{A+B} = e^{-[A,B]/2} e^A e^B$$

Hey, keen! How do I prove that?

> You don't. You thank Baker, Campbell, and Hausdorff for proving it.

Okay, thanks! (They all read the newgroups? I've seen a post from Galileo, so maybe.)

Well, *that* makes short work of this half of the problem. Let's set:

$$A = za^*/\sqrt{2}$$
$$B = -z^*a/\sqrt{2}$$

$$[A, B] = -\frac{1}{2} zz^*[a^*, a] = zz^*/2$$

which is a number and so commutes with everything, so

$$e^{A+B} = e^{-zz^*/4} e^A e^B$$

so

$$\mathrm{Coh}(z) = e^{-|z|^2/4} e^{za^*/\sqrt{2}} e^{-z^* a/\sqrt{2}} |0\rangle = e^{-|z|^2/4} e^{za^*/\sqrt{2}} |0\rangle$$

so the factor ι is 1.

Whew! Heavy firepower, just to determine that measly little phase factor ι! But then, rumor has it that Gauss spent two years of Sundays just trying to determine the sign of a certain square root.

18 John Baez: Messing around with the vacuum energy, eh?

Michael Weiss complained:

> Oh professor, I think you took too much off!

Oh, sorry. So you replaced the Hamiltonian

$$\frac{p^2 + q^2}{2} = \frac{aa^* + a^* a}{2}$$

by its normal-ordered form, where all annihilation operators are pushed to the right:

$$\frac{p^2 + q^2 - 1}{2} = a^* a$$

Somehow I hadn't noticed that. This will come in handy in full-fledged quantum field theory, but it's not necessary here, and it's sort of enlightening *not* to do it.

> Hmm, maybe you're saying the formulas are trying to tell me something—that I *shouldn't* monkey around with the zero-point?

Messing around with the vacuum energy, eh? You may unleash powerful forces — forces that mankind was never meant to meddle with! For example, you have a perfectly nice representation of the Lie algebra of the symplectic group on your hands; getting rid of the vacuum energy will turn it into a nasty *projective* representation. But never mind, go ahead, just don't blame *me* for what happens...

In simpler terms: there are lots of interesting classical observables built using quadratic expressions in the p's and q's, of which the Hamiltonian is one. When we replace the classical p's and q's by operators, we'd like Poisson brackets to go over to commutators. If we try to do this for general polynomials in the p's and q's, it doesn't work very well. However, for quadratic expressions in the p's and q's it does, *if* we don't mess with them by normal-ordering.

As for the rest of your post...

Great. You got a very nice formula:

$$\mathrm{Coh}(z) = e^{-|z|^2/4} e^{za^*/\sqrt{2}} |0\rangle$$

implying

$$\mathrm{Coh}(z) = e^{-|z|^2/4} \sum_n \frac{(z/\sqrt{2})^n}{\sqrt{n!}} |n\rangle$$

which is a very precise way of stating what you noted quite a while ago: *the number of quanta in a coherent state is given by a Poisson distribution*.

But now let's see what happens if we evolve our coherent state in time. We'll see something nice, a cute relation between the harmonic oscillator and the spin-$\frac{1}{2}$ particle, which we discussed once upon a time...

19 Michael Weiss: Dangerous signs of normality

> So you replaced the Hamiltonian [...] by its normal-ordered form

So *that's* what I was doing. Though actually I had already noted dangerous signs of normality in my thought processes, when trying to prove the Baker-Campbell-Hausdorff formula. After all, if we set $A = za^*/\sqrt{2}$, $B = -z^* a/\sqrt{2}$, then $e^A e^B$ is the normally ordered form of e^{A+B}.

Speaking of Baker-Campbell-Hausdorff: *is* there a slick proof of their formula, or does one just have to fight it out with Taylor series? I did notice one thing: we can prove pretty easily that $e^{A+B} e^{-B} e^{-A}$ is a *number* (i.e., a multiple of the identity operator) in the case at hand. First we show that this product of exponentials commutes with A, using the "derivative = bracket" trick:

$$[A, e^{A+B}e^{-B}e^{-A}] = 0$$

I omit the details (which are not messy), but it's essential here that $[A, B]$ commutes with A and B. Next we appeal to Schur's lemma. The irreducibility hypothesis is satisfied since $|0\rangle, A|0\rangle, \ldots, A^n|0\rangle, \ldots$ span the whole Hilbert space.

20 Michael Weiss: Vacuum energy, source of all of humanity's future energy needs

OK, time evolution. We have:

$$\mathrm{Coh}(z) = e^{-|z|^2/4} e^{za^*/\sqrt{2}}|0\rangle$$

and

$$H = \frac{aa^* + a^*a}{2} = a^*a + \frac{1}{2}$$

including the vacuum energy this time, source of all of humanity's future energy needs. (Right.)

We want to know how $\mathrm{Coh}(z)$ evolves (in the Schrödinger picture, since it *doesn't* evolve in the Heisenberg picture). I.e., we want to compute

$$e^{-iHt}\mathrm{Coh}(z)$$

Now it would be nice if we could just plug in the formulas we just obtained. Alas, we'd need to compute $e^{a^*a}e^{za^*}$ (give or take an annoying factor), and the Baker-Campbell-Hausdorff formula we have only helps with $e^A e^B$ when $[A, B]$ commutes with A and B. Is that the case here?

$$[a^*a, a^*] = [a^*, a^*]a + a^*[a, a^*] = 0 + a^* = a^*.$$

No such luck, the commutator does *not* commute with a^*a.

Probably we could do something slick anyway, but at least the dull way is quick.

$$\mathrm{Coh}(z) = e^{-|z|^2/4} \sum_n \frac{(z/\sqrt{2})^n}{\sqrt{n!}}|n\rangle$$

$$e^{-iHt}|n\rangle = e^{-it/2}e^{-int}|n\rangle$$

so

$$e^{-iHt}\mathrm{Coh}(z) = e^{-|z|^2/4}e^{-it/2}\sum_n \frac{(z/\sqrt{2})^n}{\sqrt{n!}}e^{-int}|n\rangle$$

but since $e^{-int} = (e^{-it})^n$, we can fold this into the $(z/\sqrt{2})^n$ factor and get:

$$e^{-iHt}\mathrm{Coh}(z) = e^{-it/2}\mathrm{Coh}(ze^{-it})$$

So after 2π seconds, we've got minus the original state vector.

21 John Baez: To quote a student of Segal…

Michael Weiss wrote:

> So after 2π seconds, we've got minus the original state vector.

Right! Just like a spin-$\frac{1}{2}$ particle, the harmonic oscillator picks up a phase of -1 when it goes all the way around. Coincidence?

Before I answer that, let me describe another way to do the computation Michael just did, which takes advantage of the relation between the Heisenberg picture and the Schrödinger picture.

In the Heisenberg picture, operators evolve in time while states stand still. After a time t, an operator A evolves to

$$A(t) = e^{iHt}Ae^{-iHt}$$

Measuring an observable $A(t)$ just means "measuring the observable A at time t".

If we differentiate this equation with respect to t we see that operators change at a rate given by their commutator with the Hamiltonian:

$$\frac{d}{dt}A(t) = i[H, A(t)] \qquad \text{(Heisenberg)}$$

Okay, so how does the creation operator evolve in time in our problem? I claim it evolves in time in the simplest possible way: just like the classical harmonic oscillator, it goes round and round!

$$a^*(t) = e^{it}a^*$$

How do we show this? Well, it's clearly true for $t = 0$, so let's just check that it satisfies (Heisenberg). First note that

$$[H, a^*] = [a^*a + 1/2, a^*] = [a^*a, a^*] = a^*$$

Its commutator with the Hamiltonian is itself! Thus we have

$$\begin{aligned} i[H, a^*(t)] &= i[H, e^{it}a^*] \\ &= ie^{it}a^* \\ &= \frac{d}{dt}e^{it}a^* \\ &= \frac{d}{dt}a^*(t) \end{aligned}$$

as desired.

So:

$$e^{iHt}a^*e^{-iHt} = e^{it}a^*$$

raise both sides to the n and you get:

$$e^{iHt}a^{*^n}e^{-iHt} = (e^{it})^n a^{*^n}$$

since adjacent factors of e^{-iHt} and e^{iHt} cancel out on the left hand side. Thus

$$e^{iHt}e^{za^*/\sqrt{2}}e^{-iHt} = \sum_n \frac{(z/\sqrt{2})^n}{n!}(e^{it})^n a^{*^n} = \exp(ze^{it}a^*/\sqrt{2})$$

Now replace t with $-t$ and move a factor to the other side, getting

$$e^{-iHt}e^{za^*/\sqrt{2}} = \exp(ze^{-it}a^*/\sqrt{2})e^{-iHt}$$

and thus

$$\begin{aligned} e^{-iHt}\mathrm{Coh}(z) &= e^{-iHt}e^{-|z|^2/4}e^{za^*/\sqrt{2}}|0\rangle \\ &= e^{-|z|^2/4}\exp(e^{-it}za^*/\sqrt{2})e^{-iHt}|0\rangle \\ &= e^{-|z|^2/4}\exp(e^{-it}za^*/\sqrt{2})e^{-it/2}|0\rangle \\ &= e^{-it/2}\mathrm{Coh}(e^{-it}z) \end{aligned}$$

The coherent state $\mathrm{Coh}(z)$ is the quantum analog of a particle at the point z in phase space. Our dot races around clockwise (with our sign conventions).

The moral is clear: when we quantize the harmonic oscillator, the creation operator evolves in a way that completely mimics the evolution of classical solutions. Since coherent states are built from the vacuum by hitting it with exponentiated creation operators, it's also true that coherent states evolve in a way which completely mimics the evolution of the corresponding classical solutions! *Except* for a phase, coming from the vacuum energy.

All this will apply to quantum field theory as well, which is why it's worthwhile going over it in such painstaking detail.

I think we are done with the harmonic oscillator! When we turn to quantum field theory, we'll find that we've already done most of the hard work.

Now, on that analogy between the harmonic oscillator and the spin-$\frac{1}{2}$ particle. . . to quote a student of Segal:

> This funny extra $\frac{1}{2}$ in the eigenvalues of the harmonic oscillator Hamiltonian can be thought of as the "zero-point energy" or "vacuum energy" due to the uncertainty principle. However, we've seen that the fact that it's exactly $\frac{1}{2}$ is no coincidence! Just as you need to give a particle of half-integer spin *two* rotations of 360 degrees for it to get back to the way it was, with no funny phase factor of -1, so you need to let the harmonic oscillator wait *two* classical periods for it to get back to exactly the way it was. In the first case we are using the fact that $SO(3)$ has a double cover $SU(2)$ — or more generally, $SO(n)$ has a double cover $Spin(n)$. In the second case we are using the fact that $Sp(2)$ has a double cover $Mp(2)$ — or more generally, $Sp(2n)$ has a double cover $Mp(2n)$.
>
> As I've said before, $SO(n)$ is to fermions as $Sp(2n)$ is to bosons. The first has to do with the canonical anticommutation relations, and Clifford algebras, while the second has to do with canonical commutation relations, and Weyl algebras. So there is a big beautiful pattern here.

22 John Baez: A lower bound on slickness

Michael Weiss wrote:

> Speaking of Baker-Campbell-Hausdorff: *is* there a slick proof of their formula, or does one just have to fight it out with Taylor series?

The Baker-Campbell-Hausdorff formula is complicated enough to set a certain lower bound on the slickness of any proof thereof. It doesn't make strong assumptions on the special nature of A and B, and it says

$$\begin{aligned} e^A e^B &= \exp(A + B + \frac{1}{2}[A, B] \\ &+ \frac{1}{12}[[A, B], B] - \frac{1}{12}[[A, B], A] \\ &- \frac{1}{48}[B, [A, [A, B]]] - \frac{1}{48}[A, [B, [A, B]]] + \ldots) \end{aligned}$$

It comes in handy you want to define the product in a Lie group knowing the bracket in the Lie algebra: it says that the latter uniquely determines the former, and says in painstaking detail exactly how.

I've never really understood the proof; I see it before me on pages 114–120 of Varadarajan's *Lie Groups, Lie Algebras, and their Representations*, but it looks like black magic.

However, we don't need the full-fledged version here, since in our example $[A, B]$ commutes with everything and all the higher terms go away. This watered-down version is a lot easier to prove.

> I did notice one thing: we can prove pretty easily that $e^{A+B}e^{-B}e^{-A}$ is a *number* . . . Next we appeal to Schur's lemma.

Watch it! Someday Schur's lemma is going to get sick of all these appeals. . . compassion fatigue will set in, and then where will we be?

If you could compute that number you'd be done. But here's a closely related approach that should also work. Assume $[A, B]$ commutes with A and B. Then to show

$$e^{A+B} = e^A e^B e^{-[A,B]/2}$$

it's obviously enough to show

$$e^{t(A+B)} = e^{tA}e^{tB}e^{-[tA,tB]/2}$$

for all real t. And for this, it suffices to show that the right-hand side satisfies the differential equation that defines the left-hand side. In other words, we just need to check that

$$\frac{d}{dt}e^{tA}e^{tB}e^{-[tA,tB]/2} = (A+B)e^{tA}e^{tB}e^{-[tA,tB]/2}$$

(If you are concerned about rigor, wave your hands and mutter the phrase "Stone's theorem" here.)

But this is just a computation! By the product rule:

$$\begin{aligned}\frac{d}{dt}e^{tA}e^{tB}e^{-[tA,tB]/2} &= Ae^{tA}e^{tB}e^{-[tA,tB]/2} \\ &+ e^{tA}Be^{tB}e^{-[tA,tB]/2} \\ &+ e^{tA}e^{tB}(-t[A,B])e^{-[tA,tB]/2}\end{aligned}$$

Then push stuff to the front and hope it all equals $A + B$. In the third term there's no problem pushing $-t[A, B]$ up to the front since it commutes with everyone. For the second term we use a fact that I think you mentioned somewhere, namely that bracketing with B acts as $[B, A]\frac{d}{dA}$ in this context, so

$$[B, e^{tA}] = t[B, A]e^{tA}$$

hence

$$e^{tA}B = Be^{tA} + t[A, B]e^{tA}$$

We get:

$$\begin{aligned}Ae^{tA}e^{tB}e^{-[tA,tB]/2} &+ \\ e^{tA}Be^{tB}e^{-[tA,tB]/2} &+ \\ e^{tA}e^{tB}(-t[A,B])e^{-[tA,tB]/2} &= \\ (A+B)e^{tA}e^{tB}e^{-[tA,tB]/2} &\end{aligned}$$

which is just what we need!

Let me decode my somewhat cryptic remarks about the drawbacks of normal-ordering. Consider *homogeneous* quadratic expressions in the classical p's and q's. If we have just one p and one q, a basis of these is given by the harmonic oscillator Hamiltonian:

$$H = \frac{p^2 + q^2}{2}$$

the kinetic energy operator:

$$K = \frac{p^2}{2}$$

and the generator of scale transformations or "dilations":

$$S = qp$$

The Poisson brackets of any two of these guys is a linear combination of these guys, so what we've got on our hands is a little 3-dimensional Lie algebra. Now, the group of symplectic transformations of a 2d phase space is $SL(2, \mathbf{R})$, so its Lie algebra is $sl(2, \mathbf{R})$, which is a 3-dimensional Lie algebra. So it's natural to guess that we've got $sl(2, \mathbf{R})$ on our hands.

Exercise: work out the Poisson brackets of H, S, and K and show they form a Lie algebra isomorphic to $sl(2, \mathbf{R})$.

Now, it turns out that if we replace the classical p's and q's in these expressions by quantum p's and q's, and pick the right factor ordering for S — carefully, because pq isn't the same as qp in quantum-land — we get operators whose commutators perfectly mimic the classical brackets. In other words, we get a representation of $sl(2, \mathbf{R})$!

This is quantization like physicists always dreamt it would be: the classical Lie algebra of symmetries is now the quantum one. Ah, were it always so simple! However, it only works if we use the above formula for H. It *doesn't* work if we use the normal-ordered version of the harmonic oscillator Hamiltonian, where we subtract off the vacuum energy. Then our commutation relations only mimic the classical Poisson brackets *up to constants*.

Exercise: show that's true.

And this is a pity, because in quantum field theory we *have to* use the normal-ordered version of the Hamiltonian. This leads to "anomalies" and other scary things.

23 Appendix: Notational Conventions

Plane wave with momentum k and energy ω: $e^{ikx-i\omega t}$.

Metric	$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2$
Momentum	$p = -id/dx$
Position	$q =$ “multiply by x”
Hamiltonian	$H = id/dt$
Annihilator	$a = (q + ip)/\sqrt{2}$
Creator	$a^* = (q - ip)/\sqrt{2}$
	$q = (a + a^*)/\sqrt{2}$
	$p = (a - a^*)/(i\sqrt{2})$
commutator	$[A, B] = AB - BA$
	$dA/dt = i[H, A]$
	$dA/dx = -i[p, A]$
product rule	$[s, AB] = [s, A]B + A[s, B]$

$$\begin{aligned}
H &= \frac{q^2 + p^2}{2} \\
aa^* &= \frac{q^2 + p^2 + 1}{2} = H + 1/2 \\
a^*a &= \frac{q^2 + p^2 - 1}{2} = H - 1/2
\end{aligned}$$

$$\begin{aligned}
[a, a^*] &= 1 \\
[a, a^{*^n}] &= na^{*^{(n-1)}} \\
[a^n, a^*] &= na^{n-1} \\
[q, p] &= i \\
[q, p^n] &= nip^{n-1} \\
[p, q] &= -i \\
[p, q^n] &= -niq^{n-1} \\
[q, e^{-icp}] &= ce^{-icp} \\
[p, e^{ibq}] &= be^{ibq}
\end{aligned}$$

a^*a is also called the number operator, sometimes denoted N.

Coherent states, take 1:

$$\mathrm{Coh1}(c + ib) = e^{ibq}e^{-icp}\,|0\rangle$$

$$
\begin{aligned}
\mathrm{Coh1}(c+ib) &= \iota e^{-(c^2+b^2)/2} \sum_n \frac{(c+ib)^n}{\sqrt{n!}} |n\rangle \\
\mathrm{Coh1}(c+ib) &= K e^{ibx-(x-c)^2}
\end{aligned}
$$

where $|\iota| = 1$, and the last equation gives Coh1 as a complex wavefunction, and K is a normalization factor.

Coherent states, final version:

$$
\begin{aligned}
\mathrm{Coh}(c+ib) &= e^{-icp+ibq}|0\rangle \\
\mathrm{Coh}(c+ib) &= e^{-(c^2+b^2)/2} \sum_n \frac{(c+ib)^n}{\sqrt{n!}} |n\rangle \\
\mathrm{Coh}(c+ib) &= K e^{ibx-(x-c)^2}
\end{aligned}
$$

If $z = c+ib$, then $e^{-iHt}\mathrm{Coh}(z) = e^{-it/2}\mathrm{Coh}(ze^{-it})$. Here $e^{-it/2}$ represents the "vacuum energy".

Baker-Campbell-Hausdorff formula, full-blown version:

$$
\begin{aligned}
e^A e^B &= \exp(A + B + \frac{1}{2}[A,B] \\
&+ \frac{1}{12}[[A,B],B] - \frac{1}{12}[[A,B],A] \\
&- \frac{1}{48}[B,[A,[A,B]]] - \frac{1}{48}[A,[B,[A,B]]] + \ldots)
\end{aligned}
$$

Baker-Campbell-Hausdorff formula, special case: if $[A, B]$ commutes with both A and B, then:

$$e^{A+B} = e^A e^B e^{-[A,B]/2}$$